B.Sc. MATHEMATICS – I YEAR

DJM1B : ALGEBRA AND SEQUENCES AND SERIES

SYLLABUS

Unit I: Theory of equation: Every equation f(x) = 0 of n^{th} degree has 'n' roots, Symmetric functions of the roots in terms of the coefficients – Sum of the r^{th} powers of the roots – Newton's theorem – Descartes rule of sign – Rolle's theorem.

Unit II: Reciprocal Equation – Transformation of equation – Solution of cubic and biquadratic equation – Cardon's land Ferrari's methods – Approximate solution of numerical equations – Newton's and Horner's methods.

Unit III: Sequence and series : Sequence – limits, bounded, monotonic, convergent, oscillatory and divergent sequence – Algebra of limits – Subsequence – Cauchy sequence in R and Cauchy's general principle of convergence.

Unit IV: Series – convergence, divergence – geometric, harmonic, exponential, binomial and logarithmic series – Cauchy's general principle of convergence – Comparison test – tests of convergence of positive termed series – Kummer's test, ratio test, Raabe's test, Cauchy's root test, Cauchy's condensation test.

Unit V: Summation of series using Binomial, Exponential and Logarithmic series.

Books for reference:

1. Algebra – Vol.I, T.K. Manickavachagompillai & others
2. Sequences and series, S. Arumugam & others
3. Real Analysis – Vol.I, K. ChandrasekaraRao & K.S. Narayanan
4. Infinite series, Bromwich.

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UNIT I: THEORY OF EQUATION

Theory of equation: Every equation f(x) = 0 of n^{th} degree has 'n' roots, Symmetric functions of the roots in terms of the coefficients – Sum of the r^{th} powers of the roots – Newton's theorem – Descartes rule of sign – Rolle's theorem.

Theory of Equations:

Every equation f(x) = 0 of the n^{th} degree has n roots

Let f(x) be the polynomial $a_0x^n + a_1x^{n-1} + \ldots + a_n$.

We assume that every equation f(x) = 0 has at least one root real or imaginary

Let α_1 be a root of f(x) = 0.

Then f(x) is exactly divisible by $x - \alpha_1$, so that

$$f(x) = (x - \alpha_1)\, \phi_1 (x)$$

where $\phi_1(x)$ is a rational integral function of degree n – 1.

Again $\phi_1 (x) = 0$ has a root real or imaginary and let that root be a_2.

Then $\phi_1 (x)$ is exactly divisible by $x - \alpha_2$, so that

$$\phi_1(x) = (x - \alpha_2)\, \phi_2 (x)$$

where $\phi_2(x)$ is a rational integral function of degree n – 2.

$$\therefore f(x) = (x - \alpha_1)(x - \alpha_2)\, \phi_2 (x).$$

By continuing in this way, we obtain

$$f(x) = (x - \alpha_1)(x - \alpha_2)\ldots (x - \alpha_n)\, \phi_n(x)$$

where $\phi_n(x)$ is of degree n – n, i.e., zero

$\therefore \phi_n(x)$ is a constant.

Equating the coefficients of x^n on both sides we get

$$\phi_n(x) = \text{coefficients of } x^n$$

$$= a_0$$

$$\therefore f(x) = a_0 (x - \alpha_1)(x - \alpha_2)\ldots (x - \alpha_n).$$

Hence the equation f(x) =0 has n roots, since f(x) vanished when x has any one of the values $\alpha_1, \alpha_2, \ldots \alpha_n$. If x is given any value different from any one of these n roots, then no factor of f(x) can vanish and the equation is not satisfied. Hence f(x) = 0 cannot have more than n roots.

Example. 1. If α be a real root of the cubic equation $x^3 + px^2 + qx + r = 0$, of which the coefficients are real, show that the other two roots of the equation are real, if

$$p^2 \geq 4q + 2p\alpha + 3\alpha^2.$$

Solution.

Since α is a root of the equation, $x^3 + px^2 + qx + r$ is exactly divisible by $x - \alpha$.

$\therefore$ Let $x^3 + px^2 + qx + r \equiv (x - \alpha)(x^2 + ax + b)$.

Equating the coefficients of powers of x on both sides, we get

$$p = -\alpha + a$$
$$q = -a\alpha + b$$
$$r = -b\alpha$$

$\therefore a = p + \alpha$ and $b = q + a\alpha = q + \alpha(p + \alpha)$

$$= q + p\alpha + \alpha^2.$$

The other two roots of the equation are the roots of

$x^2 + (p + \alpha)x + q + p\alpha + \alpha^2 = 0$

Which are real if $(p + \alpha)^2 - 4(q + p\alpha + \alpha^2) \geq 0$

i.e., $p^2 - 2p\alpha - 4q - 3\alpha^2 \geq 0$

i.e., $p^2 \geq 4q + 2p\alpha + 3\alpha^2$.

Example 2. If $x_1, x_2, x_3 \ldots x_n$ are the roots of the equation $(a_1 - x)(a_2 - x)\ldots(a_n - x) + k = 0$, then show that $a_1, a_2, \ldots, a_n$ are the roots of the equation

$$(x_1 - x)(x_2 - x)\ldots(x_n - x) - k = 0.$$

Solution.

Since $x_1, x_2, x_3 \ldots x_n$ are the roots of the equation

$$(a_1 - x)(a_2 - x)\ldots(a_n - x) + k = 0$$

We have

$(a_1 - x)(a_2 - x)\ldots(a_n - x) + k \equiv (x_1 - x)(x_2 - x)\ldots(x_n - x)$

$\therefore (x_1 - x)(x_2 - x)\ldots(x_n - x) - k \equiv (a_1 - x)(a_2 - x)\ldots(a_n - x)$.

$\therefore a_1, a_2, a_3 \ldots a_n$ are the roots of

$(x_1 - x)(x_2 - x)\ldots(x_n - x) - k = 0$.

Example. 3. Show that if *a*, *b*, *c* are real, the roots of

$\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=\frac{3}{x}$ are real.

Solution.

Simplifying we get

$$x(x+b)(x+c)+x(x+c)(x+a)+x(x+a)(x+b)$$
$$-3(x+a)(x+b)(x+c)=0$$

Let f(x) be the expression on the left-hand side. It can easily be seen that f(x) is a quadratic function of .

$$\therefore f(-a) = -a(b-a)(c-a)$$
$$f(-b) = -b(c-b)(a-b)$$
$$f(-c) = -c(a-c)(b-c).$$

Without loss of generally let us assume that a> b> c and a, b, c are all positive.

Then $a-b$, $b-c$, $a-c$ are positive.

$\therefore$ $f(-a) = -$ ve.

$f(-b) = +$ ve.

$f(-c) = -$ ve.

$\therefore$ The equation has at least one real root between $-a$ and $-b$, and another between $-b$ and $-c$.

The equation can have only two roots since $f(x) = 0$ is a quadratic equation.

$\therefore$ The roots of the equations are real.

Exercises

1. If $x^3 + 3px + q$ has a factor of the form $x^2 - 2ax + a^2$, show that $q^2 + 4p^3 = 0$.
2. If $px^3 + qx + r$ has a factor of the form $x^2 + ax + 1$, prove that $p^2 = pq + r^2$.
3. If $px^5 + qx^2 + r$ has a factor of the form $x^2 + ax + 1$, prove that

   $(p^2 - r^2)(p^2 - r^2 + qr) = p^2q^2$.
4. If a, b, c are all positive, show that all the roots of

$$\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = \frac{1}{x}$$ are real.

5. If $a > b > c > d$ and E, A, B, C, D are positive, show that the equation

$$E + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{x-d} = 0$$

has no root between *a* and *b*, one root between *b* and *c* and one between *c* and *d* and if

$E > 0$, there is *a* root $> d$ and if $E < 0$, there is a root $< a$.

6. If $a < b < c < d$, show that the roots of $(x - a)(x - c) = k(x - b)(x - d)$ are real for all values of k.

In an equation with rational coefficients, imaginary roots occur in pairs.

Let the equation be f(x) = 0 and let $\alpha + i\beta$ be an imaginary root of the equation. We shall show that $\alpha - i\beta$ is also a root.

We have $(x - \alpha - i\beta)(x - \alpha + i\beta) = (x - \alpha)^2 + \beta^2$ ………(1)

If f(x) is divided by $(x - \alpha)^2 + \beta^2$, let the quotient be Q(x) and the remainder be Rx + R'

Here Q(x) is of degree (n – 2).

$\therefore f(x) = \{(x - \alpha)^2 + \beta^2\} Q(x) + Rx + R'$ ………(2)

Substituting $(\alpha + i\beta)$ for x in the equation (2), we get

$f(\alpha + i\beta) = \{(\alpha + i\beta - \alpha)^2 + \beta^2\} Q(\alpha + i\beta) + R(\alpha + i\beta) + R'$

$= R(\alpha + i\beta) + R'$

But $f(\alpha + i\beta) = 0$ since $\alpha + i\beta$ is a root of f(x) = 0.

Therefore

$R(\alpha + i\beta) + R' = 0.$

Equating to zero the real and imaginary parts

$R\alpha + R' = 0$ and $R\beta = 0.$

Since $\beta \neq 0$, R = 0 and so R' = 0

$\therefore f(x) = \{(x-\alpha)^2 + \beta^2\}Q(x)$.

$\therefore \alpha - i\beta$ is also a root of f(x) = 0.

Solved Problems

1. Form a rational cubic equation which shall have for roots 1, $3 - \sqrt{-2}$.

Solution.

Since $3 - \sqrt{-2}$ is a root of the equation, $3 + \sqrt{-2}$ is also a root. So we have to form an equation whose roots are 1, $3 - \sqrt{-2}$, $3 + \sqrt{-2}$.

Hence the required equation is $(x-1)(x-3-\sqrt{-2})(x-3+\sqrt{-2}) = 0$

$$(x-1)\{(x-3)^2 + 2\} = 0$$

$$(x-1)(x^2 - 6x + 11) = 0$$

$$x^3 - 7x^2 + 17x - 11 = 0.$$

2. Solve the equation $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$ of which one root is $-1 + \sqrt{-1}$.

Solution.

Imaginary roots occur in pairs. Hence $-1 - \sqrt{-1}$ is also a root of the equation.

Therefore the expression on the left side of equation has the factors

$(x + 1 - \sqrt{-1})(x + 1 + \sqrt{-1})$.

The expression on the left side is exactly divisible by $(x+1)^2 + 1$, i.e., $x^2 + 2x + 2$.

Dividing $x^4 + 4x^3 + 5x^2 + 2x - 2$ by $x^2 + 2x + 2$, we get the quotient $x^2 + 2x - 1$.

Therefore $x^4 + 4x^3 + 5x^2 + 2x - 2 = (x^2 + 2x + 2)(x^2 + 2x - 1)$.

Hence the other roots are obtained from $x^2 + 2x - 1 = 0$.

Thus the other roots are $-1 \pm \sqrt{2}$.

3. Show that $\frac{a^2}{x-\alpha}+\frac{b^2}{x-\beta}+\frac{c^2}{x-\gamma}-x+\delta=0$ has only real roots if a, b, c, α, β, γ, δ are real.

Solution.

If possible let p + iq be a root. Then p – iq is also root.

Substituting these values for x, we have

$$\frac{a^2}{p+iq-\alpha}+\frac{b^2}{p+iq-\beta}+\frac{c^2}{p+iq-\gamma}-p-iq+\delta=0 \quad ......(1)$$

$$\frac{a^2}{p-iq-\alpha}+\frac{b^2}{p-iq-\beta}+\frac{c^2}{p-iq-\gamma}-p+iq+\delta=0 \quad ......(2)$$

Substituting (2) from (1), we get

$$-\frac{2a^2iq}{(p-\alpha)^2+q^2}-\frac{2b^2iq}{(p-\beta)^2+q^2}-\frac{2c^2iq}{(p-\gamma)^2+q^2}-2iq=0$$

$$-2iq\left\{\frac{a^2}{(p-\alpha)^2+q^2}+\frac{b^2}{(p-\beta)^2+q^2}+\frac{c^2}{(p-\gamma)^2+q^2}+1\right\}=0$$

This is only possible when q = 0 since the other factor cannot be zero. In that case the roots are real.

In an equation with rational coefficients irrational roots occur in pairs.

Let f(x) = 0 denotes the equation and suppose that a + $\sqrt{b}$ is a root of the equation where a and b are rational and $\sqrt{b}$ is irrational. We now show that a – $\sqrt{b}$ is also a root of the equation

$$(x-a-\sqrt{b})(x-a+\sqrt{b})=(x-a)^2-b \quad ....(1)$$

If f(x) is divided by $(x-a)^2-b$, let the quotient be Q(x) and the remainder be Rx + R'.

Here Q(x) is a polynomial of degree (n – 2).

$\therefore f(x)=\{(x-a)^2-b\}\,Q(x)+Rx+R'$ ....(2)

Substituting a + $\sqrt{b}$ for x in (2), we get

$f(a + \sqrt{b}) = \{(a + \sqrt{b} – a)^2 – b\}\ Q(a + \sqrt{b}) + R(a + \sqrt{b}) + R'$

$= R(a + \sqrt{b}) + R'$

but $f(a + \sqrt{b}) = 0$, since $a + \sqrt{b}$ is a root of $f(x) = 0$.

$\therefore\ Ra + R' + R\sqrt{b} = 0.$

Equating the rational and irrational parts, we have

$Ra + R' = 0$ and $R = 0$.

$\therefore R' = 0.$

Hence $f(x) = \{(x – a)^2 – b\}Q(x)$.

$= (x – a – \sqrt{b})(x – a + \sqrt{b})Q(x)$.

$\therefore a – \sqrt{b}$ is a root of $f(x) = 0$.

Solved Problems

Example 1. Frame an equation with rational coefficients, one of whose root is $\sqrt{5} + \sqrt{2}$

Solution.

Then the other roots are $\sqrt{5} - \sqrt{2}, -\sqrt{5} + \sqrt{2}, -\sqrt{5} - \sqrt{2}$

Hence the required equation is $(x-\sqrt{5} - \sqrt{2})(x -\sqrt{5} + \sqrt{2})(x +\sqrt{5} + \sqrt{2})(x +\sqrt{5} - \sqrt{2}) = 0$

i.e. $\{(x – \sqrt{5})^2 – 2\}\{(x +\sqrt{5})^2 – 2\} = 0$

i.e. $(x^2 – 2x\sqrt{5} +3)(x^2 + 2x\sqrt{5} +3) = 0$

i.e. $(x^2 + 3)^2 – 4x^2.5 = 0$

i.e. $x^4 – 14x^2 + 9 = 0$.

Example 2. Solve the equation $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$ given that one of the roots is $1 - \sqrt{5}$.

Solution.

Since the irrational roots occur in pairs, $1 + \sqrt{5}$ is also a root. The factors corresponding to these roots are

$(x - 1 + \sqrt{5})(x - 1 - \sqrt{5})$, i.e.$(x - 1)^2 - 5$

i.e. $x^2 - 2x - 4$.

Dividing $x^4 - 5x^3 + 4x^2 + 8x - 8$ by $x^2 - 2x - 4$, we get the quotient $x^2 - 3x + 2$.

Therefore $x^4 - 5x^3 + 4x^2 + 8x - 8 = (x^2 - 2x - 4)(x^2 - 3x + 2)$

$= (x^2 - 2x - 4)(x - 1)(x - 2)$

The roots of the equation are $1 \pm \sqrt{5}$, 1, 2.

Example 3. Form the equation with rational coefficients whose roots are

(i) $1 + 5\sqrt{-1}, 5 - \sqrt{-1}$

(ii) $-\sqrt{3} + \sqrt{-2}$.

Solution :

(i) $1 + 5\sqrt{-1}, 5 - \sqrt{-1}$

Then the other roots are $1 + 5\sqrt{-1}, 5 - \sqrt{-1}$, $1 - 5\sqrt{-1}, 5 + \sqrt{-1}$

Hence the equation is

$(x - 1 + 5\sqrt{-1})(x - 1 - 5\sqrt{-1})(x - 5 - \sqrt{-1})(x - 5 + \sqrt{-1}) = 0$

$\{(x - 1)^2 - (5\sqrt{-1})^2\}\left\{(x-5)^2 - \left(\sqrt{-1}\right)^2\right\} = 0$

$(x^2 - 2x + 26)(x^2 - 10x + 26) = 0$

$x^4 - 12x^3 + 72x^2 - 312x + 676 = 0.$

(ii) $-\sqrt{3} + \sqrt{-2}$

Then the other roots are $-\sqrt{3} + \sqrt{-2}, -\sqrt{3} - \sqrt{-2}, \sqrt{3} + \sqrt{-2}, \sqrt{3} - \sqrt{-2}$

$$\left\{\left(x + \sqrt{3}\right)^2 - \left(\sqrt{-2}\right)^2\right\}\left\{\left(x - \sqrt{3}\right)^2 - \left(\sqrt{-2}\right)^2\right\} = 0$$

$(x^2 + 2\sqrt{3}x + 5)(x^2 - 2\sqrt{3}x + 5) = 0$

$x^4 - 2x^2 + 25 = 0.$

Example 4. Solve : $x^4 - 4x^3 + 8x + 35 = 0$ given that $2 + i\sqrt{3}$ is a root of it.

Solution.

Since the irrational roots occur in pair, $2 - i\sqrt{3}$ is also a root.

The factors corresponding to these roots are $(x - 2)^2 - (i\sqrt{3})^2$

$x^2 - 4x + 7.$

Dividing $x^4 - 4x^3 + 8x + 35$ by $x^2 - 4x + 7$, we get the equation $x^2 + 4x + 5$

$x^4 - 4x^3 + 8x + 35 = (x^2 - 4x + 7)(x^2 + 4x + 5)$

The roots of the equation are $2 \pm i\sqrt{3}, -2 \pm i$

Example 5.Solve the equation $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81 = 0$ given that one root is $\sqrt{2} - \sqrt{-1}$.

Solution.

Then the other roots are $\sqrt{2} - \sqrt{-1}, \sqrt{2} + \sqrt{-1}, -\sqrt{2} - \sqrt{-1}, -\sqrt{2} + \sqrt{-1}$

$$\left\{\left(x - \sqrt{2}\right)^2 - \left(\sqrt{-1}\right)^2\right\}\left\{\left(x + \sqrt{2}\right)^2 - \left(\sqrt{-1}\right)^2\right\} = 0$$

$(x^2 - 2\sqrt{2}x + 3)(x^2 + 2\sqrt{2}x + 3) = 0$

$x^4 - 2x^2 + 9 = 0$

Dividing $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81$ by $x^4 - 2x^2 + 9$ we get the equation $2x^2 - 3x + 9$

$2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81 = (x^4 - 2x^2 + 9)(2x^2 - 3x + 9)$

The roots of the equation are $\sqrt{2} \pm \sqrt{-1}$, , $-\sqrt{2} \pm \sqrt{-1}$, $3\left(\frac{1 \pm i\sqrt{7}}{4}\right)$

Exercises

1. Find the equation with rational coefficients whose roots are
   (i) $4\sqrt{3}, 5 + 2\sqrt{-1}$.
   (ii) $\sqrt{-1} - \sqrt{5}$.

2. Solve the equation $x^4 + 2x^3 - 5x^2 + 6x + 2 = 0$ given that $1 + \sqrt{-1}$ is a root of it
3. Solve the equation $x^6 - 4x^5 - 11x^4 + 40x^3 + 11x^2 - 4x - 1 = 0$ given that one root is $\sqrt{2} - \sqrt{3}$.

Answer : 1. (i) $x^4 - 10x^3 - 19x^2 + 480x - 1392 = 0$, (ii) $x^4 - 8x^2 + 36 = 0$, 2. $-2 \pm \sqrt{3}$, 3. $2 \pm \sqrt{3}$, $2 \pm \sqrt{5}$.

Relation between the roots and coefficient of equations.

Let the equation be $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0$. If this equation has the roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, then we have

$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$

$= (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$

$= x^n - \sum \alpha_1 x^{n-1} + \sum \alpha_1\alpha_2 x^{n-2} - \ldots + (-1)^n \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$

$= x^n - S_1x^{n-1} + S_2x^{n-2} - \ldots + (-1)^nS_n$

Where S_r is the sum of the products of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ taken r at a time.

Equating the coefficients of like powers on both sides, we have

$-p_1 = S_1$ = sum of the roots.

$(-1)^2p_2 = S_2$ = sum of the products of the roots taken two at a time.

$(-1)^3p_3 = S_3$ = sum of the products of the roots taken three at a time.

$(-1)^np_n = S_n$ = product of the roots.

If the equation is $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$.

Divide each term of the equation by a_0.

The equation becomes $x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0$

and so we have

$$\sum \alpha_1 = -\frac{a_1}{a_0}$$

$$\sum \alpha_1\alpha_2 = \frac{a_2}{a_0}$$

$$\sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

$$\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

These n equations are of no help in the general solution of an equation but they are often helpful in the solution of numerical equations when some special relation is known to exist among the roots. The method is illustrated in the examples given below.

Example 1. Show that the roots of the equation $x^3 + px^2 + qx + r = 0$ are in Arithmetical progression if $2p^3 - 9pq + 27r = 0$ show that the above condition is satisfied by the equation $x^3 - 6x^2 + 13x - 10 = 0$. Hence or otherwise solve the equation.

Solution.

Let the roots of the equation $x^3 + px^2 + qx + r = 0$ be $\alpha - \delta, \alpha, \alpha + \delta$.

We have from the relation of the roots and coefficients

$$\alpha - \delta + \alpha + \alpha + \delta = -p$$

$$(\alpha - \delta)\alpha + (\alpha - \delta)(\alpha + \delta) + \alpha(\alpha + \delta) = q$$

$$(\alpha - \delta)\alpha(\alpha + \delta) = -r.$$

Simplifying these equation, we get

$$3\alpha = -p \qquad \ldots(1)$$

$$3\alpha^2 - \delta^2 = q \qquad \ldots(2)$$

$$\alpha^3 - \alpha\delta^2 = -r. \qquad \ldots(3)$$

From (1), $\alpha = -\frac{p}{3}$.

From (2), $\delta^2 = 3\left(-\frac{p}{3}\right)^2 - q = \frac{p^2}{3} - q$.

Substituting these value in (3), we get

$$\left(-\frac{p}{3}\right)^3 - \left(-\frac{p}{3}\right)\left(\frac{p^2}{3} - q\right) = -r$$

i.e., $2p^3 - 9pq + 27r = 0$.

In the equation $x^3 - 6x^2 + 13x - 10 = 0$.

$p = -6, q = 13, r = -10$.

Therefore $2p^3 - 9pq + 27r = 2(-6)^3 - 9(-6)13 + 27(-10) = 0$

The condition is satisfied and so the roots of the equation are in arithmetical progression. In this case the equations (1), (2), (3) become

$3\alpha = 6$

$3\alpha^2 - \delta^2 = 13$

$\alpha^3 - \alpha\delta^2 = 10.$

$\alpha = 2,\ 12 - \delta^2 = 13$

Therefore $\delta^2 = -1$

i.e., $\delta = \pm i$.

The roots are $2 - i, 2, 2 + i$.

Example 2. Find the condition that the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may be in geometric progression. Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in geometric progression.

Solution.

Let the roots of the equation be $\frac{k}{r}$, k, kr.

Therefore $\frac{k}{r} + k + kr = -\frac{3b}{a}$ …..(1)

$\frac{k^2}{r} + k^2 + k^2 r = \frac{3c}{a}$ ….(2)

$k^3 = -\frac{d}{a}$ …..(3)

From (1), $k\left(\frac{1}{r} + 1 + r\right) = -\frac{3b}{a}$.

From (2), $k^2\left(\frac{1}{r} + 1 + r\right) = \frac{3c}{a}$.

Divided one by the other, we get $k = -\frac{c}{b}$

Substituting this value of k in (3), we get $\left(-\frac{c}{b}\right)^3 = -\frac{d}{a}$.

Therefore $ac^3 = b^3 d$.

In the equation $27x^3 + 42x^2 - 28x - 8 = 0$

$\frac{k}{r} + k + kr = -\frac{42}{27}$

$\frac{k^2}{r} + k^2 + k^2 r = -\frac{28}{27}$

$k^3 = \frac{8}{27}$

$\therefore\ k = \frac{2}{3}$.

Substituting the value of k in(4), we get

$\frac{2}{3}\left(\frac{1}{r} + 1 + r\right) = -\frac{42}{27}$

$3r^2 + 10r + 3 = 0$

$(3r + 1)(r + 3) = 0$

Therefore $r = -\frac{1}{3}$ or $r = -3$.

For both the value of r, the roots are $-2, \frac{2}{3}, -\frac{2}{9}$.

Example 3. Solve the equation $81x^3 - 18x^2 - 36x + 8 = 0$ whose roots are in harmonic progression.

Solution.

Let the roots be α, β, γ.

Then $\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$

i.e., $2\gamma\alpha = \beta\gamma + \alpha\beta$ ……(1)

From the relation between the coefficients and the roots we have

$$\alpha + \beta + \gamma = \frac{18}{81} \quad \text{…….(2)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{36}{81} \quad \text{……..(3)}$$

$$\alpha\ \beta\ \gamma = -\frac{8}{81} \quad \text{….(4)}$$

From (1) and (3), we get

$$2\gamma\alpha + \gamma\alpha = -\frac{36}{81}$$

$$3\gamma\alpha = -\frac{36}{81}$$

Therefore $\gamma\alpha = -\frac{4}{27}$ …….(5)

Substituting this value of γα in (4), we get

$$\beta\left(-\frac{4}{27}\right) = -\frac{8}{81}$$

Therefore $\beta = \frac{2}{3}$.

From (2), we have

$$\alpha + \gamma = \frac{18}{81} - \frac{2}{3} = -\frac{4}{9} \quad \text{……(6)}$$

From (5)and (6), we get

$\alpha = \frac{2}{9}$ and $\gamma = -\frac{2}{3}$

The roots are $\frac{2}{9}, \frac{2}{3}$ and $-\frac{2}{3}$.

Example 4. If the sum of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two, prove that $p^3 + 8r = 4pq$.

Solution.

Let the roots of the equation be α, β, γ and δ

Then $\alpha + \beta = \gamma + \delta$ …..(1)

From the relation of the coefficients and the roots, we have

$\alpha + \beta + \gamma + \delta = -p$ ………(2)

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$ ………..(3)

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$ ………..(4)

$\alpha\beta\gamma\delta = s$ ………..(5)

From (1) and (2), we get

$2(\alpha + \beta) = -p$ ………(6)

(3) can be written as

$\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = q$

i.e., $(\alpha\beta + \gamma\delta) + (\alpha + \beta)^2 = q$ ……(7)

(4) can be written as

$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -r$

$(\alpha\beta + \gamma\delta)(\alpha + \beta) = -r$ ……(8)

From (6) and (7), we get

$\alpha\beta + \gamma\delta + \frac{p^2}{4} = q$

$\therefore \quad \alpha\beta + \gamma\delta = q - \frac{p^2}{4}$ …..(9)

From (8), we get

$$-\frac{p}{2}(\alpha\beta + \gamma\delta) = -r$$

$$\alpha\beta + \gamma\delta = \frac{2r}{p} \qquad \ldots\ldots\ (10)$$

Equating (9) and (10), we get

$$q - \frac{p^2}{4} = \frac{2r}{p}$$

$$4pq - p^3 = 8r$$

$$p^3 + 8r = 4pq.$$

Example 5. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ given that two of its roots are equal in magnitude and opposite in sign.

Solution.

Let the roots of the equation be α, β, γ and δ

Here $\gamma = -\delta$

i.e., $\gamma + \delta = 0 \qquad \ldots\ldots(1)$

From the relation of the roots and coefficients

$$\alpha + \beta + \gamma + \delta = 2 \qquad \ldots\ldots(2)$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 4 \qquad \ldots\ldots(3)$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -6 \qquad \ldots\ldots(4)$$

$$\alpha\beta\gamma\delta = -21 \qquad \ldots\ldots(5)$$

from (1) and (2), we get $\alpha + \beta = 2 \qquad \ldots\ldots(6)$

(3) can be written as $\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = 4$

$$\alpha\beta + \gamma\delta = 4 \qquad \ldots\ldots(7)$$

(4) can be written as $\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -6$

$\gamma\delta(\alpha + \beta) = -6$ ....(8)

from (6) and (8), we get $\gamma\delta = -3$ .....(9)

but $\gamma + \delta = 0$ $\therefore$ $\gamma = \sqrt{3}$, $\delta = -\sqrt{3}$.

From (7) and (9), we get $\alpha\beta = 7$

$\therefore$ α and β are the roots of x2 – 2x + 7 = 0.

$\therefore$ $\alpha = 1 + \sqrt{-6}$, $\beta = 1 - \sqrt{-6}$

Therefore the roots of the equation are $\pm\sqrt{3}$, $1 \pm\sqrt{-6}$.

Example 6. Find the condition that the general bi quadratic equation $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ may have two pairs of equal roots.

Solution.

Let the roots be $\alpha, \alpha, \beta, \beta$.

From the relations of coefficients and roots

$$2\alpha + 2\beta = -\frac{4b}{a} \quad \text{................(1)}$$

$$\alpha^2 + \beta^2 + 4\alpha\beta = \frac{6c}{a} \quad \text{............(2)}$$

$$2\alpha\beta^2 + 2\alpha^2\beta = -\frac{4d}{a} \quad \text{................(3)}$$

$$\alpha^2\beta^2 = \frac{e}{a} \quad \text{............(4)}$$

From (1), we get $\alpha + \beta = -\frac{2b}{a}$ ....(5)

From (3), we get $2\alpha\beta(\alpha + \beta) = -\frac{4d}{a}$

$$\therefore \alpha\beta = \frac{d}{b} \quad \text{.........(6)}$$

From (5) and (6), we get that α, β are the roots of the equation $x^2 + \frac{2b}{a}x + \frac{d}{b} = 0$

$$\therefore ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv a\left(x^2 + \frac{2b}{a}x + \frac{d}{b}\right)^2$$

Comparing coefficients

$$6c = a\left(\frac{4b^2}{a^2} + \frac{2d}{b}\right) \text{ and } e = \frac{ad^2}{b^2}$$

$\therefore 3abc = a^2d + 2b^3$ and $eb^2 = ad^2$.

Exercises

1. Solve the equation $6x^3 - 11x^2 + 6x - 1 = 0$ whose roots are in harmonic progression.
2. Find the values of a and b for which the roots of the equation $4x^4 - 16x^3 + ax^2 + bx - 7 = 0$ are in arithmetical progression.
3. The roots of the equation $8x^3 - 14x^2 + 7x - 1 = 0$ are in geometrical progression. Find them.
4. Solve $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$, it being given that the sum of two of the roots is equal to the sum of the other two.
5. If two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ are equal in value but differ in sign, show that $r^2 + p^2s = pqr$.
6. Show that the four roots, α, β, γ and δ of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ will be connected by the relation $\alpha\beta + \gamma\delta = 0$ if $p^2s + r^2 = 4qs$.
7. Solve the equation $x^4 - 2x^3 - 3x^2 + 4x - 1 = 0$given that the product of two of the roots is unity.

   Answer : 1.1, $\frac{1}{2}, \frac{1}{3}$, 2.a = 4 or $-\frac{4}{9}$, b = 24 or $\frac{296}{9}$, 3. $\frac{1}{4}, \frac{1}{2}$, 1, 4. – 1, 5, 1, 3, 7. $\frac{3\pm\sqrt{5}}{2}, \frac{-1\pm\sqrt{5}}{2}$

Symmetric function of the roots

If a function involving all the roots of an equation is unaltered in value if any two of the roots are interchanged, it is called a symmetric function of the roots.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_n$ be the roots of the equation.

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots. + p_n = 0.$$

We have learned that

$$S_1 = \Sigma\, \alpha_1 = -\, p_1$$

$$S_2 = \Sigma\, \alpha_1\alpha_2 = p_2$$

$S_3 = \Sigma \alpha_1 \alpha_2 \alpha_3 = -p_3$

……………………..

……………………..

Without knowing the values of the roots separately in terms of the coefficients, by using the above relations between the coefficients and the roots of an equation, we can express any symmetric function of the roots in terms of the coefficients of the equations.

Example 1. If α, β, γ are the roots of the equations $x^3 + px^2 + qx + r = 0$, Express the value of $\Sigma \alpha^2 \beta$ in terms of the coefficients.

Solution.

We have $\alpha + \beta + \gamma = -p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -r.$

$\Sigma \alpha^2\beta = \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma$

$= q(-p) - 3(-r)$

$= 3r - pq.$

Example 2. If α, β, γ, δ be the roots of the bi quadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$, Find (1) $\Sigma \alpha^2$, (2) $\Sigma \alpha^2 \beta\gamma$, (3) $\Sigma \alpha^2 \beta^2$, (4) $\Sigma \alpha^3 \beta$ and (5) $\Sigma \alpha^4$.

Solution.

The relations between the roots and the coefficients are

$\alpha + \beta + \gamma + \delta = -p.$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$

$\alpha\beta\gamma\delta = s.$

$\Sigma\, \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$= (\alpha + \beta + \gamma + \delta)^2 - 2\, (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$= (\Sigma\, \alpha)^2 - 2\, \Sigma\, \alpha\beta$

$= p^2 - 2q.$

$\Sigma\, \alpha^2\, \beta\gamma = (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)\, (\alpha + \beta + \gamma + \delta) - 4\, \alpha\beta\gamma\delta$

$= (\Sigma\, \alpha\beta\gamma)\, (\Sigma\, \alpha) - 4\, \alpha\beta\gamma\delta$

$= pr - 4s.$

$\Sigma\, \alpha^2\, \beta^2 = \alpha^2\, \beta^2 + \alpha^2\, \gamma^2 + \alpha^2\, \delta^2 + \beta^2\, \gamma^2 + \beta^2\, \delta^2 + \gamma^2\, \delta^2$

$= (\Sigma\, \alpha\beta)^2 - 2\, \Sigma\, \alpha^2\, \beta\gamma - 6\, \alpha\beta\gamma\delta$

$= q^2 - 2\, (pr - 4s) - 6s$

$= q^2 - 2pr + 2s.$

$\Sigma\, \alpha^3\, \beta = (\Sigma\, \alpha^2)\, (\Sigma\, \alpha\beta) - \Sigma\, \alpha^2\, \beta\gamma$

$= (p^2 - 2q)\, q - (pr - 4s)$

$= p^2q - 2q^2 - pr + 4s.$

$\Sigma\, \alpha^4 = (\Sigma\, \alpha^2)^2 - 2\, \Sigma\, \alpha^2\, \beta^2$

$= (p^2 - 2q)^2 - 2\, (q^2 - 2pr + 2s)$

$= p^4 - 4p^2q + 2q^2 + 4pr - 4s.$

Example 3. If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, from the equation whose roots are $\alpha\beta$, $\beta\gamma$, and $\gamma\alpha$.

Solution.

The relations between the roots and coefficients are

$$\alpha + \beta + \gamma = -a$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = b$$

$$\alpha\beta\gamma = -c.$$

The required equation is

$$(x - \alpha\beta)(x - \beta\gamma)(x - \gamma\alpha) = 0$$

i.e., $x^3 - x^2(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2)x - \alpha^2\beta^2\gamma^2 = 0$

i.e., $x^3 - x^2(\alpha\beta + \beta\gamma + \gamma\alpha) + x\,\alpha\beta\gamma(\alpha + \beta + \gamma) - (\alpha\beta\gamma)^2 = 0$

i.e., $x^3 - bx^2 + acx - c^2 = 0$

Example 4. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, from the equation whose roots are $\beta + \gamma - 2\alpha$, $\gamma + \alpha - 2\beta$, $\alpha + \beta - 2\gamma$.

Solution.

We have $\alpha + \beta + \gamma = -p$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$\alpha\beta\gamma = -r.$$

In the required equation

S_1 = Sum of the roots = $\beta + \gamma - 2\alpha + \gamma + \alpha - 2\beta + \alpha + \beta - 2\gamma$

= 0.

S_2 = Sum of the products of the roots taken two at a time

$= (\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta) + (\beta + \gamma - 2\alpha)(\alpha + \beta - 2\gamma) + (\alpha + \beta - 2\gamma)(\gamma + \alpha - 2\beta)$

$= (\alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 3\beta)$ + 2 similar terms

$= (-p - 3\alpha)(-p - 3\beta) + (-p - 3\alpha)(-p - 3\gamma) + (-p - 3\gamma)(-p - 3\beta)$

$= (p + 3\alpha)(p + 3\beta) + (p + 3\alpha)(p + 3\gamma) + (p + 3\gamma)(p + 3\beta)$

$= 3p^2 + 6p(\alpha + \beta + \gamma) + 9(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 3p^2 + 6p(-p) + 9q$

$= 9q - 3p^2.$

S_3 = Products of the roots

$= (\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta)(\alpha + \beta - 2\gamma)$

$= (\alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 3\beta)(\alpha + \beta + \gamma - 3\gamma)$

$= (-p - 3\alpha)(-p - 3\beta)(-p - 3\gamma)$

$= -\{ p^3 + 3p^2(\alpha + \beta + \gamma) + 9p(\alpha\beta + \beta\gamma + \gamma\alpha) + 27\alpha\beta\gamma \}$

$= -\{ p^3 + 3p^2(-p) + 9pq - 27r \}$

$= 2p^2 - 9pq + 27r$

Hence the required equation is

$x^3 - S_1x^2 + S_2x - S_3 = 0$

i.e., $x^3 + (9q - 3p^2)x - (2p^3 - 9pq + 27r) = 0.$

Example 5. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ prove that

(1) $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = r - pq$

(2) $\alpha^3 + \beta^3 + \gamma^3 = -p^3 + 3pq - 3r.$

Solution.

We have $\alpha + \beta + \gamma = -p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -r.$

(1). $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = [-(p + \alpha)(p + \beta)(p + \gamma)]$

Since α + β + γ = – p ∴ α + β = – p – γ

$= -[p^3 + p^2(\alpha+\beta+\gamma) + p(\alpha\beta+\beta\gamma+\gamma\alpha) + \alpha\beta\gamma]$

$= -[p^3 + p^2 \times -p + pq - r] = -[p^3 - p^3 + pq - r] = r - pq.$

(2). $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha+\beta+\gamma)[\alpha^2 + \beta^2 + \gamma^2 - (\alpha\beta+\beta\gamma+\gamma\alpha)]$

$\sum\alpha^3 = \sum\alpha\,[\sum\alpha^2 - \sum\alpha\beta] + 3\alpha\beta\gamma;$

But $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

Therefore $\sum\alpha^3 = \sum\alpha\,[(\sum\alpha)^2 - 3\sum\alpha\beta] + 3\alpha\beta\gamma$; = – p[p2 – 3q] – 3r = – p3 + 3pq – 3r.

Example 6. If α, β, γ are the roots of the equation $x^3 + qx + r = 0$ find the values of

(1) $\sum\frac{1}{\beta+\gamma}$.

(2) $\sum\frac{\beta^2+\gamma^2}{\beta+\gamma}$

Solution.

Since α, β, γ are the roots of the equation $x^3 + qx + r = 0$.

We have α + β + γ = 0

αβ + βγ + γα = q

αβγ = – r.

Therefore β + γ = – α

(1). $\sum\frac{1}{\beta+\gamma} = \sum\frac{1}{-\alpha} = -\left[\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right] = -\frac{\sum\alpha\beta}{\alpha\beta\gamma} = \frac{-q}{-r} = \frac{q}{r}$

(2). $\sum\frac{\beta^2+\gamma^2}{\beta+\gamma} = \sum\frac{(\beta+\gamma)^2-2\beta\gamma}{\beta+\gamma} = \frac{\sum\left[\alpha^2+2\frac{r}{\alpha}\right]}{-\alpha} = \frac{\sum\alpha^3+2r}{-\alpha^2} = -\sum\alpha - 2\sum\frac{r}{\alpha^2}$

$= -2r\sum\frac{1}{\alpha^2}$; since $\sum\alpha = 0$

But $\sum\frac{1}{\alpha^2} = \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{\alpha^2\beta^2\gamma^2} = \frac{(\sum\alpha\beta)^2}{(\alpha\beta\gamma)^2}$ since $(\alpha\beta+\beta\gamma+\gamma\alpha)^2 = \sum\alpha^2\beta^2 + 2\alpha\beta\gamma\sum\alpha = \sum\alpha^2\beta^2$; since $\sum\alpha = 0$

$\sum \alpha^2\beta^2$ = q^2 ;$\sum \frac{1}{\alpha^2} = \frac{q^2}{r^2} = \frac{q^2}{r^2}$

$\therefore \sum \frac{\beta^2+\gamma^2}{\beta+\gamma} = \frac{-2q^2r}{r^2} = \frac{-2q^2}{r}$.

Example 7. If α, β, γ are the roots of the equation $x^3 - px^2 + qx - r = 0$ find the value of

(1). $\sum \frac{\beta^2+\gamma^2}{\beta\gamma}$

(2). $\sum(\beta+\gamma-\alpha)^2$.

Solution.

Since α, β, γ are the roots of the equation $x^3 - px^2 + qx - r = 0$

We have $\alpha + \beta + \gamma = p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = r$.

(1). $\sum \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\alpha^2+\beta^2}{\alpha\beta} + \frac{\alpha^2+\gamma^2}{\alpha\gamma} = \frac{\alpha(\beta^2+\gamma^2)+\gamma(\alpha^2+\beta^2)+\beta(\alpha^2+\gamma^2)}{\alpha\beta\gamma}$

$= \frac{\sum \alpha^2\beta}{\alpha\beta\gamma}$

But $\sum \alpha^2\beta = (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma$

$\frac{\sum \alpha^2\beta}{\alpha\beta\gamma} = \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma}{\alpha\beta\gamma} = \frac{qp-3r}{r}$.

(2). $\sum(\beta+\gamma-\alpha)^2 = \sum(\alpha+\beta+\gamma-2\alpha)^2 = \sum(p-2\alpha)^2 = \sum(p^2+4\alpha^2-4\alpha\beta)$

$= 3p^2 + 4\sum \alpha^2 - 4p\sum \alpha\beta$

$= 3p^2 + 4\left[\left(\sum \alpha\right)^2 - 2\sum \alpha\beta\right] - 4p^2$

$= 3p^2 + 4p^2 - 8q - 4p^2$

$= 3p^2 - 8q$.

Example 8. If α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$find the value of

$$\Sigma \frac{1}{\alpha^2\beta^2}$$

Solution.

Since α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$

We have $\alpha + \beta + \gamma = \frac{-b}{a}$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

$$\Sigma \frac{1}{\alpha^2\beta^2} = \frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha^2\beta^2\gamma^2} = \frac{(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)}{(\alpha\beta\gamma)^2} = \frac{\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)}{\left(\frac{d}{a}\right)^2}$$

$$= \frac{b^2 - 2ac}{d^2}$$

Exercises

1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ find the value of

   (1) $(\beta + \gamma - \alpha)^3 + (\gamma + \alpha - \beta)^3 + (\alpha + \beta - \gamma)^3$.

   (2) $\frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta}$.

2. If α, β, γ, δ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$,

   Evaluate (1) $\Sigma \alpha^2\beta\gamma$, (2) $\Sigma (\beta + \alpha + \delta)^2$ and (3) $\Sigma \frac{1}{\alpha^2}$.

   Answer : 1. (1).$24r - p^3$, (2).$\frac{2rp - q^2}{r}$, 2. (1).$pr - 4s$, (2).$3p^2 - 2q$,(3).$\frac{r^2 - 2qr}{s}$

Sum of the powers of the roots of an equation.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ be the roots of an equation f (x) = 0.The sum of the r^{th} powers of the roots

i.e., $\alpha_1^r + \alpha_1^r + \ldots\ldots + \alpha_n^r$

is usually denoted by S_r. We can easily see that S_r constitutes a symmetric function of the roots and hence we can calculate the value of S_r by the methods described in the previous article. When r is greater than 4, the calculation of S_r by the previous method becomes tedious and in those cases, the following two methods can be used profitably.

We have $f(x) = (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n)$.

Taking logarithms on both sides and differentiating, we get

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \ldots\ldots + \frac{1}{x-\alpha_n}$$

$$\frac{xf'(x)}{f(x)} = \frac{x}{x-\alpha_1} + \frac{x}{x-\alpha_2} + \ldots\ldots + \frac{x}{x-\alpha_n}$$

$$= \frac{1}{1-\frac{\alpha_1}{x}} + \frac{1}{1-\frac{\alpha_2}{x}} + \ldots\ldots + \frac{1}{1-\frac{\alpha_n}{x}}$$

$$= \left(1 - \frac{\alpha_1}{x}\right)^{-1} + \left(1 - \frac{\alpha_2}{x}\right)^{-1} + \ldots\ldots + \left(1 - \frac{\alpha_n}{x}\right)^{-1}$$

$$= 1 + \frac{\alpha_1}{x} + \frac{\alpha_1^2}{x^2} + \ldots. + \frac{\alpha_1^n}{x^n} + \ldots$$

$$+ \; 1 + \frac{\alpha_2}{x} + \frac{\alpha_2^2}{x^2} + \ldots. + \frac{\alpha_2^n}{x^n} + \ldots$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

$$+ \; 1 + \frac{\alpha^n}{x} + \frac{\alpha_n^2}{x^2} + \ldots. + \frac{\alpha_n^n}{x^n} + \ldots$$

$$= n + (\Sigma\, \alpha_1)\frac{1}{x} + (\Sigma\, \alpha_1^2)\frac{1}{x^2} + \ldots.. + (\Sigma\, \alpha_1^r)\frac{1}{x^r} + \ldots.$$

$$= n + S_1 \cdot \frac{1}{x} + S_2 \cdot \frac{1}{x^2} + \ldots. + S_r \cdot \frac{1}{x^r} + \ldots.$$

$\therefore$ S_r = Coefficient of $\frac{1}{x^r}$ in the expansion of $\frac{xf'(x)}{f(x)}$.

Example. Find the sum of the cubes of the roots of the equation $x^5 = x^2 + x + 1$.

Solution.

The equation can be written in the form

$$f(x) = x^5 - x^2 - x - 1 = 0$$

S_r = Coefficient of $\frac{1}{x^3}$ in the expansion of $\frac{x\,(5x^4 - 2x - 1)}{x^5 - x^2 - x - 1}$

= Coefficient of $\frac{1}{x^3}$ in $\frac{5 - \frac{2}{x^3} - \frac{1}{x^4}}{1 - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5}}$

= " " $\left(5 - \frac{2}{x^3} - \frac{1}{x^4}\right)\left(1 - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5}\right)^{-1}$

= " " $\left(5 - \frac{2}{x^3} - \frac{1}{x^4}\right)\left\{1 + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \left(\frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5}\right)^2 + \right.$ ...

= " " $\left(5 - \frac{2}{x^3} - \frac{1}{x^4}\right)\left(1 + \frac{1}{x^3} + \ldots\right)$

= 3.

Newton's Theorem on the sum of the powers of the roots.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ be the roots of an equation

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots p_n = 0$$

and let be $S_r = \alpha_1^r + \alpha_2^r + \ldots\ldots + \alpha_n^r$ so that $S_0 = n$.

$$f(x) = (x - \alpha_1)(x - \alpha_2)\ldots\ldots(x - \alpha_n).$$

Taking logarithms on both sides and differentiating, we get

$$\frac{f'(x)}{f(x)} = \frac{1}{x - \alpha_1} + \frac{1}{x - \alpha_2} + \ldots\ldots + \frac{1}{x - \alpha_n}$$

i.e., $f'(x) = \frac{f(x)}{x - \alpha_1} + \frac{f(x)}{x - \alpha_2} + \ldots\ldots + \frac{f(x)}{x - \alpha_n}$

By actual division, we obtain

$$\frac{f(x)}{x-\alpha_1} = x^{n-1} + (\alpha_1 + p_1)\, x^{n-2} + (\alpha_1^2 + p_1\alpha_1 + p_2)\, x^{n-3}$$

$$+ \ldots\ldots (\alpha_1^{n-1} + p_1\alpha_1^{n-2} + \ldots + p_{n-1})$$

$$\frac{f(x)}{x-\alpha_1} = x^{n-1} + (\alpha_2 + p_1)\, x^{n-2} + (\alpha_2^2 + p_1\alpha_2 + p_2)\, x^{n-3}$$

$$+ \ldots\ldots (\alpha_2^{n-1} + p_1\alpha_2^{n-2} + \ldots + p_{n-1})$$

……………………………………………………………

$$\frac{f(x)}{x-\alpha_n} = x^{n-1} + (\alpha_n + p_1)\, x^{n-2} + (\alpha_n^2 + p_1\alpha_n + p_2)\, x^{n-3}$$

$$+ \ldots\ldots (\alpha_n^{n-1} + p_1\alpha_n^{n-2} + \ldots + p_{n-1}).$$

Adding all these functions, we get

$$f'(x) = nx^{n-1} + (S_1 + np_1)x^{n-2} + (S_2 + p_1S_1 + np_2)x^{n-3}$$

$$+ \ldots\ldots (S_{n-1} + p_1S_{n-2} + \ldots\ np_{n-1}).$$

But $f'(x)$ is also equal to

$$nx^{n-1} + (n-1)\, p_1x^{n-2} + (n-2)\, p_2x^{n-3} + \ldots + 2p_{n-2} + p_{n-1}.$$

Equating the coefficients in two values of $f'(x)$, we obtain the following relations :

$$S_1 + p_1 = 0$$

$$S_2 + p_1S_1 + 2p_2 = 0$$

$$S_3 + p_1S_2 + p_2S_1 + 3p_3 = 0$$

$$S_4 + p_1S_3 + p_2S_2 + p_3S_1 + 4p_4 = 0$$

……………………………………

……………………………………

$S_{n-1} + p_1S_{n-2} + p_2S_{n-3} + \ldots + p_{n-2}S_1 + (n-1)\, p_{n-1} = 0$

From these (n – 1) relations we can calculate in succession the values of $S_1, S_2, S_3, \ldots S_{n-1}$ in terms of the coefficients $p_1, p_2, p_3, \ldots p_{n-1}$. We can extend our results to the sums of all positive powers of the roots, viz., $S_n, S_{n+1}, \ldots S_r$ where r > n.

We have $x^{r-n} f(x) = x^r + p_1x^{r-1} + p_2x^{r-2} + \ldots + p_nx^{r-n}$.

Replacing in this identity, x by the roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, in succession and adding, we have

$$S_r + p_1S_{r-1} + p_2S_{r-2} + \ldots + p_nS_{r-n} = 0$$

Now giving r the values n, n +1, n +2,…… successively and observing that $S_0 = n$, we obtain from the last equation

$$S_n + p_1S_{n-1} + p_2S_{n-2} + \ldots + np_n = 0$$

$$S_{n+1} + p_1S_n + p_2S_{n-1} + \ldots + p_nS_1 = 0$$

$$S_{n+2} + p_1S_{n+1} + p_2S_n + \ldots + p_nS_2 = 0$$

and so on.

Thus we get

$$S_r + p_1S_{r-1} + p_2S_{r-2} + \ldots + rp_r = 0, \text{ if } r < n$$

And $S_r + p_1S_{r-1} + p_2S_{r-2} + \ldots p_nS_{r-n} = 0$, if $r \geq n$.

Cor. To find the sum of the negative integral powers of the roots of f (x) = 0, put $x = \frac{1}{y}$ and find the sums of the corresponding positive powers of the roots of the transformed equation.

Example 1. Show that the sum of the eleventh powers of the roots of $x^7 + 5x^4 + 1 = 0$ is zero.

Solution.

Since 11 is greater than 7, the degree of the equation, we have to use the latter equation in Newton's theorem.

If we assume the equation as

$x^7 + p_1x^6 + p_2x^5 + p_3x^4 + p_4x^3 + p_5x^2 + p_6x + p_7 = 0$,

we have $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 0$, $p_3 = 5$, $p_7 = 1$.

$\therefore\ S_{11} + p_1S_{10} + p_2S_9 + p_3S_8 + p_4S_7 + p_5S_6 + p_6S_5 + p_7S_4 = 0$

i.e., $S_{11} + 5S_8 + S_4 = 0$ ……(1)

Again

$S_8 + p_1S_7 + p_2S_6 + p_3S_5 + p_4S_4 + p_5S_3 + p_6S_2 + p_7S_1 = 0$

i.e., $S_8 + 5S_5 + S_1 = 0$ ……(2)

Using the first equation in the Newton's theorem

$S_5 + p_1S_4 + p_2S_3 + p_3S_2 + p_4S_1 + 5p_5 = 0$

i.e., $S_5 + 5S_2 = 0$ ……(3)

Again

$S_4 + p_1S_3 + p_2S_2 + p_3S_1 + 4p_4 = 0$

i.e., $S_4 + 5S_1 = 0$ ……(4)

Again

$S_2 + p_1S_1 + 2p_2 = 0$

i.e., $S_2 = 0$ ……..(5)

Also $S_1 = 0$ ……..(6)

From (4) , (5) and (6) , we get $S_4 = 0$

From (3) , (5) , we get $S_5 = 0$

From (2), we get $S_8 = 0$

Substituting the values of S_4 , S_8 in (1) , we get $S_{11} = 0$.

Example 2. If a + b + c + d = 0, show that

$$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \cdot \frac{a^3+b^3+c^3+d^3}{3}$$

Solution.

Since a + b + c + d = 0, we can consider that a, b, c, d are the roots of the equation

$x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$ where $p_1 = 0$.

From Newton's theorem on the sums of powers of the roots, we get

$$S_5 + p_1S_4 + p_2S_3 + p_3S_2 + p_4S_1 = 0 \quad \text{........(1)}$$

$$S_4 + p_1S_3 + p_2S_2 + p_3S_1 + 4p_4 = 0 \quad \text{......(2)}$$

$$S_3 + p_1S_2 + p_2S_1 + 3p_3 = 0 \quad \text{.......(3)}$$

$$S_2 + p_1S_1 + 2p_2 = 0 \quad \text{.........(4)}$$

$$S_1 + p_1 = 0 \quad \text{.........(5)}$$

From (5), we get $S_1 = 0$

From (4), we get $S_2 = -2p_2$

From (3), we get $S_3 = -3p_3$

From (1), we get $S_5 - 3p_2p_3 - 2p_3p_2 = 0$

i.e., $S_5 = 5p_2p_3$.

$$\therefore \frac{S_5}{5} = \frac{S_2}{2} \cdot \frac{S_3}{3}$$

i.e., $$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \cdot \frac{a^3+b^3+c^3+d^3}{3} .$$

Example 3. Find $\frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5}$ where α , β , γ are the roots of the equation

$x^3 + 2x^2 - 3x - 1 = 0.$

Solution.

Put $x = \frac{1}{y}$ in the equation, then the equation becomes

$$\frac{1}{y^3} + \frac{2}{y^2} - \frac{3}{y} - 1 = 0$$

i.e., $y^3 + 3y^2 - 2y - 1 = 0$

The roots of the equation are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

$$\therefore \frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5} = S_5 \text{ for the equation } y^3 + 3y^2 - 2y - 1 = 0.$$

From Newton's theorem on the sum of the powers of the roots of the equations, we get

$$S_5 + 3S_4 - 2S_3 - S_2 = 0$$

$$S_4 + 3S_3 - 2S_2 - S_1 = 0$$

$$S_3 + 3S_2 - 2S_1 - S_0 = 0$$

$$S_2 + 3S_1 - 4 = 0$$

$$S_1 + 3 = 0.$$

$\therefore$ $S_1 = -3$, $S_2 = 13$, $S_3 = -42$, $S_4 = 149$, $S_5 = -518$.

$$\therefore \frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5} = -518.$$

Example 4. Show that the sum of the m^{th} powers, where $m \leq n$, of the roots of the equation

$x^n - 2x^{n-1} - 2x^{n-2} - \ldots\ldots\ldots - 2x - 2 = 0$ is $3^m - 1$.

Solution.

If m ≤ n, we get from the Newton's theorem

$S_m - 2\,S_{m-1} - 2\,S_{m-2} - \ldots - m\,.\,2 = 0$

$S_{m-1} - 2\,S_{m-2} - \ldots - (m-1)\,2 = 0.$

Subtracting one from another, we get

$S_m - 3\,S_{m-1} - 2 = 0$

i.e., $S_m = 2 + 3\,S_{m-1}$

$= 2 + 3\,(2 + 3\,S_{m-2})$

$= 2 + 3.2 + 3^2\,S_{m-2}$

$= 2 + 3.2 + 3^2\,(2 + 3\,S_{m-3})$

$= 2 + 3.2 + 3^2\,.2 + 3^3\,S_{m-3}.$

Continuing like this, we get

$S_m = 2 + 3.2 + 3^2.\,2 + 3^3.\,2 + \ldots + 3^{m-1}.\,S_1$

But $S_1 = 2$.

$\therefore\ S_m = 2 + 3.2 + 3^2.\,2 + 3^3.\,2 + \ldots + 3^{m-1}.\,2$

$= 2\,(1 + 3 + 3^2 + 3^3 + \ldots + 3^{m-1})$

$= 2\,.\,\frac{(3^m - 1)}{2}$

$= 3^m - 1$

Example 5. Determine the value of $\phi(\alpha_1) + \phi(\alpha_2) + \ldots + \phi(\alpha_n)$

Where $\alpha_1, \alpha_2, \alpha_3, \ldots\ \alpha_n$ are the roots of f (x) and $\phi(x)$ is any rational and integral function of x.

Solution.

We have $\frac{f'(x)}{f(x)} = \frac{1}{x - \alpha_1} + \frac{1}{x - \alpha_2} + \ldots\ldots + \frac{1}{x - \alpha_n}$

and $\frac{f'(x)\,\phi(x)}{f(x)} = \frac{\phi(x)}{x - \alpha_1} + \frac{\phi(x)}{x - \alpha_2} + \ldots\ldots + \frac{\phi(x)}{x - \alpha_n}$

Performing the division and retaining only the remainders on both sides of the equation, we have

$$\frac{R_0 x^{n-1} + R_1 x^{n-2} + \ldots + R_{n-1}}{f(x)} = \frac{\phi(\alpha_1)}{x - \alpha_1} + \frac{\phi(\alpha_2)}{x - \alpha_2} + \ldots\ldots + \frac{\phi(\alpha_n)}{x - \alpha_n}.$$

Hence

$$R_0 x^{n-1} + R_1 x^{n-2} + \ldots + R_{n-1} = \Sigma\, \phi(\alpha_1)(x - \alpha_2)\ldots(x - \alpha_n).$$

Equating the coefficients of x^{n-1} on both sides of the equation,

We get $\Sigma\, \phi(\alpha_1) = R_0$.

Example 6. If the degree of $\phi(x)$ does not exceed n -2, prove that $\sum_1^n \frac{\phi(\alpha_r)}{f'(\alpha_1)} = 0$.

Solution.

We have partial functions

$$\frac{\phi(x)}{f(x)} = \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \ldots + \frac{A_n}{x - \alpha_n}$$

$$\therefore\ \phi(x) = A_1(x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_n) + A_2(x - \alpha_1)(x - \alpha_3)\ldots(x - \alpha_n) + \ldots$$

$$+ A_n(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_{n-1}).$$

Put $x = \alpha_1$. $\therefore\ \phi(\alpha_1) = A_1(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\ldots(\alpha_1 - \alpha_n)$.

$$f(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n).$$

$$f'(x) = (x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_n) + (x - \alpha_1)(x - \alpha_3)\ldots(x - \alpha_n)$$

$$+ \ldots\ldots\ldots + (x - \alpha_1)(x - \alpha_2) \ldots . (x - \alpha_{n-1}).$$

$$\therefore \quad f'(\alpha_1) = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \ldots . (\alpha_1 - \alpha_n).$$

$$\therefore \quad \phi(\alpha_1) = A_1 \, f'(\alpha_1)$$

i.e., $A_1 = \dfrac{\phi(\alpha_1)}{f'(\alpha_1)}$.

Hence $\dfrac{\phi(x)}{f(x)} = \dfrac{\phi(\alpha_1)}{f'(\alpha_1)} \cdot \dfrac{1}{x - \alpha_1} + \dfrac{\phi(\alpha_2)}{f'(\alpha_2)} \cdot \dfrac{1}{x - \alpha_2} + \cdots + \dfrac{\phi(\alpha_n)}{f'(\alpha_n)} \cdot \dfrac{1}{x - \alpha_n}$

$$= \sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f'(\alpha_r)} \cdot \frac{1}{x - \alpha_r} .$$

$$\therefore \quad \frac{x\,\phi(x)}{f(x)} = \sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f'(\alpha_r)} \cdot \frac{x}{x - \alpha_r} .$$

$$= \sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f'(\alpha_r)} \cdot \frac{1}{(1 - \frac{\alpha_r}{x})} .$$

$$= \sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f'(\alpha_r)} \left\{ 1 + \frac{\alpha_r}{x} + \left(\frac{\alpha_r}{x}\right)^2 + \cdots \right\}.$$

$\sum_{r=1}^{n} \dfrac{\phi(\alpha_r)}{f'(\alpha_r)} =$ term independent of x in $\dfrac{x\,\phi(x)}{f(x)}$.

$\phi(x)$ is of degree n – 2 , f (x) is of degree n.

Hence $x\,\phi(x)$ is of degree n – 1.

$$\therefore \quad \frac{x\,\phi(x)}{f(x)} = \frac{B_0 x^{n-1} + B_1 x^{n-2} + \cdots . + B_{n-1}}{x^n + p_1 x^{n-1} + \cdots . + p_n}$$

$$= \frac{\frac{B_0}{x} + \frac{B_1}{x^2} + \cdots .. \frac{B_{n-1}}{x^n}}{1 + \frac{p_1}{x} + \frac{p_2}{x^2} + \cdots .. \frac{p_n}{x^n}}$$

Hence in the expansion of $\dfrac{x\,\phi(x)}{f(x)}$ there is no term independent of x.

$$\therefore \quad \sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f'(\alpha_r)} = 0.$$

Exercises

1. Show that the sum of the fourth powers of the roots of the equation
   $x^5 + px^3 + qx^2 + s = 0$ is $2p^2$.
2. If α, β, γ are the roots of $x^3 + qx + r = 0$ prove that

   (1) $3s_2 s_5 = 5s_3 s_4$.

   (2) $\frac{\alpha^5+\beta^5+\gamma^5}{5} = \frac{\alpha^3+\beta^3+\gamma^3}{3} \cdot \frac{\alpha^2+\beta^2+\gamma^2}{2}$.

   (3) $\frac{\alpha^7+\beta^7+\gamma^7}{7} = \frac{\alpha^5+\beta^5+\gamma^5}{5} \cdot \frac{\alpha^2+\beta^2+\gamma^2}{2}$.
3. Show that the sum of the ninth powers of the roots of $x^3 + 3x + 9 = 0$ is zero.
4. Prove that the sum of the twentieth powers of the roots of the equation
   $x^4 + ax + b = 0$ is $50\, a^4b^2 - 4b^5$.

Descartes' Rule of signs.

An equation f (x) = 0 cannot have more positive roots than there are changes of sign in f (x)

Let f(x) be a polynomial whose signs of the terms are

+ + - - - + - + + + - + -.

In this there are seven changes of sign including changes from + to – and from – to +. We shall show that if this polynomial be multiplied by a binomial (corresponding to a positive root) whose signs of the terms are + – , the resulting polynomial will have atleast one more change of sign than the original. Writing down only the signs of the terms in the multiplication, we have

```
+ + – – – + – + + + – + –
                        + –
---------------------------
  – – + + + – + – – – + – +
+ + – – – + – + + + – + –
---------------------------
+ ± – ± ± + – + ± ± – + – +
```

Here in the last line the ambiguous sign $\pm$ is placed wherever there are two different signs to be added. Here we see in the product

(1) An ambiguity replaces each continuation of sign in the original polynomial.
(2) The sign before and after an ambiguity or a set of ambiguities are unlike and
(3) A change of sign is introduced in the end.

Let us take the most unfavourable case and suppose that all the ambiguities are replaced by continuations, then the sign of the terms become

+ + – – – + – + + + – + – +

The number of changes of sign is 8. Thus even in the most unfavourable case there is one more change of sign than the number of changes of sign in the original polynomial. Therefore we may conclude in general that the effect of multiplication of a binomial factor x – α is to introduce at least one change of sign.

Suppose the product of all the factors corresponding to negative and imaginary roots of f(x) = 0 be a polynomial F(x). The effect of multiplying F(x) by each of the factors x – α, x – β, x – γ,….. corresponding to the positive roots, α, β, γ is to introduce at least one change of sign for each, so that when the complete product is formed containing all the roots, we have the resulting polynomial which has at least as many changes of signs as it has positive roots. This is **Descartes' rule** of sign.

Descartes' rule of signs for negative roots.

Let f (x) = (x – α_1) (x – α_2)…… (x – α_n).

By subtracting – x instead of x in the equations, we get

f (– x) = (– x – α_1) (– x – α_2)…… (– x – α_n).

The roots of f (– x) = 0 are – α_1, – α_2, …. , – α_n.

∴ The negative roots of f(x) = 0 become the positive roots of f(– x) = 0.

Hence to find the maximum number of negative roots of f(x) = 0 , it is enough to find the maximum number of positive roots of f(– x) = 0.

So we can enunciate Descartes' rule for negative roots as follows.

No equation can have a greater number of negative roots then there are changes of sign in the terms of the polynomial f (–x).

Example. Determine completely the nature of the roots of the equation $x^5 - 6x^2 - 4x + 5 = 0$.

Solution.

The series of signs of the terms are + – – +.

Here there are two changes of sign.

Hence there cannot be more than two positive roots.

Changing x into –x , the equation becomes

$$-x^5 - 6x^2 + 4x + 5 = 0$$

i.e., $x^5 + 6x^2 - 4x - 5 = 0$.

The series of the signs of the terms are

+ + – –.

Here there is only one change of sign.

∴ There cannot be more than one negative root.

So the equation has got at the most three real roots. The total number of roots of the equation is 5. Hence there are at least two imaginary roots of the equation. We can also determine the limits between which the real roots lie.

$x =$	$-\infty$	-2	-1	0	1	2	∞
$x^5 - 6x^2 - 4x + 5 =$	–	–	+	+	–	+	+

The positive roots lie between 0 and 1 , and 1 and 2 , the negative root between –2 and –1.

Exercises

1. Show that the equation $x^7 - 3x^4 - 2x^3 - 1 = 0$ has at least four imaginary roots.
2. Show that $x^6 + 3x^2 - 5x + 1 = 0$ has at least four imaginary roots.
3. Prove that the equation $x^4 + 3x - 1 = 0$ has two real and two imaginary roots.
4. Show that $12x^7 - x^4 + 10x^3 - 28 = 0$ has at least four imaginary roots.

Rolles' Theorem.

Between two consecutive real roots a and b of the equation f (x) = 0 where f (x) is a polynomial , there lies at least one real root of the equation $f'(x) = 0$.

Let f (x) be $(x-a)^m (x-b)^m \phi(x)$ where *m* and *n* are positive integers and $\phi(x)$ is not divisible by (x – a) or by (x – b). since *a* and *b* are consecutive real roots of f (x), the sign of $\phi(x)$ in the interval $a \le x \le b$ is either positive throughout or negative throughout, for if it changes its sign between *a* and *b*, then there is a root of $\phi(x) = 0$ that is of f (x) = 0 lying between *a* and *b*, which is contrary to the hypothesis that *a* and *b* are consecutive roots.

$$\therefore \quad f'(x) = (x-a)^m n(x-b)^{n-1} \phi(x) + m(x-a)^{m-1}(x-b)^n \phi(x) + (x-a)^m (x-b)^n \phi'(x)$$

$$= (x-a)^{m-1} (x-b)^{n-1} \chi(x),$$

Where $\chi(x) = \{m(x-b) + n(x-a)\}\phi(x) + (x-a)(x-b)\phi'(x)$.

$$\therefore \quad \chi(a) = m(a-b)\phi(a)$$

$$\chi(b) = n(b-a)\phi(b).$$

$\chi(a)$ and $\chi(b)$ have different signs since $\phi(a)$ and $\phi(b)$ have the same sign.

$\therefore \quad \chi(x) = 0$ has atleast one root between *a* and *b*.

Hence $f'(x) = 0$ has at least one root between *a* and *b*.

Cor. 1. If all the roots of f (x) = 0 are real , then all the roots of $f'(x) = 0$ are also real.

If f (x) = 0 is a polynomial of degree *n* , $f'(x) = 0$ is a polynomial of degree n – 1 and each root of $f'(x) = 0$ lies in each of the (n – 1) intervals between the *n* roots of f (x) = 0.

Cor. 2. If all the roots of f (x) = 0 are real , then all the roots of $f'(x) = 0$, $f''(x) = 0$, $f'''(x) = 0$ are real.

Cor. 3. At the most only one real root of f (x) = 0 can lie between two consecutive roots of $f'(x) = 0$, that is the real roots of $f'(x) = 0$ separate those of f (x) = 0.

Cor. 4. If $f'(x) = 0$ has *r* real roots, then f (x) = 0 cannot have more than (*r* +1) real roots.

Cor. 5. f (x) = 0 has at least as many imaginary roots as $f'(x) = 0$.

Example 1. Find the nature of the roots of the equation $4x^3 - 21x^2 + 18x + 20 = 0$.

Solution.

Let us consider the function $f(x) = 4x^3 - 21x^2 + 18x + 20$.

We have $f'(x) = 12x^2 - 42x + 18$

$= 6(2x - 1)(x - 3)$.

Hence the real roots of $f'(x) = 0$ are $\frac{1}{2}$ and 3. So the roots of f (x) = 0, if any will be in the intervals between $-\infty$ and $\frac{1}{2}$, $\frac{1}{2}$ and 3, 3 and $+\infty$ respectively.

x :	$-\infty$	$\frac{1}{2}$	3	∞
f (x) :	–	+	–	+

$\therefore$ f (x) must vanish , once in each of the above intervals.

Hence f (x) = 0 has three real roots.

Example 2. Show that the equation $3x^4 - 8x^3 - 6x^2 + 24x - 7 = 0$ has no positive, one negative and two imaginary roots.

Solution.

Let f (x) be $3x^4 - 8x^3 - 6x^2 + 24x - 7$.

We have $f'(x) = 12x^3 - 24x^2 - 12x - 24$

$= 12(x+1)(x - 1)(x - 2)$.

The roots of $f'(x) = 0$ are -1, $+1$, $+2$.

x :	$-\infty$	-1	$+1$	$+2$	$+\infty$
f (x) :	+	–	+	+	+

∴ f (x) = 0 has a real root lying between – 1 and – ∞, one between – 1 and + 1 and two imaginary roots.

We know that f (+ 1) = + , f (0) = – .

∴ The real root lying between – 1 and + 1 lies between 0 and + 1 . hence it is a positive root. The other real root lies between – 1 and – ∞ and so it is a negative root.

Example 3. Discuss the reality of the roots $x^4 + 4x^3 - 2x^2 - 12x + a = 0$ for all the values of a.

Solution.

Let f (x) be $x^4 + 4x^3 - 2x^2 - 12x + a$.

∴ $f'(x) = 4x^3 + 12x^2 - 4x - 12$

$= 4(x + 1)(x - 1)(x + 3)$.

∴ The roots of $f'(x) = 0$ are – 3 , – 1 and 1.

x :	– ∞	– 3	– 1	1	+ ∞
f (x) :	+	a – 9	7+ a	a – 9	+

If a – 9 is negative and 7 + a is positive , the four roots of f (x) are real.

∴ If $-7 < a < 9$, f (x) = 0 has four real roots.

If a > 9 , then f (x) is positive throughout and hence all the rots of f (x) = 0 are imaginary.

If a < – 7 , the sign of f (x) at – ∞ , – 3 , – 1 , 1 , + ∞ are respectively + , – , – , – , +.

Hence f (x) = 0 has two real roots and two imaginary roots.

Exercises

1. Prove that all the roots of the equation $x^3 - 18x + 25 = 0$ are real.
2. Find the nature of the roots of the equation

   (1) $4x^3 - 21x^2 + 18x + 30 = 0$.

(2) $2x^3 - 9x^2 + 12x + 3 = 0$.

(3) $x^4 + 4x^3 - 20x^2 + 10 = 0$.

3. Show that the equation $f(x) = (x - a)^3 + (x - b)^3 + (x - c)^3 = 0$ has one real and two imaginary roots.

Answer : 2.(1). One negative root and two imaginary roots, (2). One negative root and two imaginary roots, (3). All the roots are real.

UNIT II: RECIPROCAL EQUATION

Reciprocal Equation – Transformation of equation – Solution of cubic and biquadratic equation – Cardon's land Ferrari's methods – Approximate solution of numerical equations – Newton's and Horner's methods.

Reciprocal roots.

To transform an equation into another whose roots are the reciprocals of the roots of the given equation.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ be the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots p_n = 0.$$

We have

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots p_n \equiv (x - \alpha_1)(x - \alpha_2)\ldots\ldots(x - \alpha_n).$$

Put $x = \frac{1}{y}$, we have

$$\left(\frac{1}{y}\right)^n + p_1\left(\frac{1}{y}\right)^{n-1} + p_2\left(\frac{1}{y}\right)^{n-2} + \ldots\ldots + p_n$$

$$= \left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)\ldots\ldots\left(\frac{1}{y} - \alpha_n\right)$$

Multiplying throughout by y^n, we have

$$p_ny^n + p_{n-1}y^{n-1} + p_{n-2}y^{n-2} + \ldots\ldots + p_1y + 1 = 0$$

$$= (\alpha_1\ \alpha_2 \ldots\ldots \alpha_n)\left(\frac{1}{\alpha_1} - y\right)\left(\frac{1}{\alpha_2} - y\right)\ldots\ldots\left(\frac{1}{\alpha_n} - y\right)$$

Hence the equation

$p_ny^n + p_{n-1}y^{n-1} + p_{n-2}y^{n-2} + \ldots\ldots + p_1y + 1 = 0$ has roots $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$

Reciprocal equation.

If an equation remains unaltered when x is changed into its reciprocal, it is called reciprocal equation.

Let $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n = 0.$ ……(1)

be a reciprocal equation. When x is changed into its reciprocal $\frac{1}{x}$, we get the transformed equation

$$p_nx^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \ldots\ldots + p_1x + 1 = 0$$

$$x^n + \frac{p_{n-1}}{p_n}x^{n-1} + \frac{p_{n-2}}{p_n}x^{n-2} + \ldots + \frac{p_1}{p_n}x + \frac{1}{p_n} = 0 \quad \ldots(2)$$

Since (1) is a reciprocal equation , it must be the same as (2),

$$\therefore \frac{p_{n-1}}{p_n} = p_1, \frac{p_{n-2}}{p_n} = p_2 \ldots \frac{p_1}{p_n} = p_{n-1} \text{ and } \frac{1}{p_n} = p_n.$$

$$\therefore p_n^2 = 1.$$

$$\therefore p_n = \pm 1.$$

Case i. $p_n = 1$.

Then $p_{n-1} = p_1, p_{n-2} = p_2, p_{n-3} = p_3, \ldots\ldots$

In this case the coefficients of the terms equidistant from the beginning and the end are equal in magnitude and have the same sign.

Case ii. $p_n = -1$, we have

$$p_{n-1} = -p_1, p_{n-2} = -p_2, \ldots\ p_1 = -p_{n-1}.$$

In this case the terms equidistant from the beginning and the end are equal in magnitude but different in sign.

Standard form of reciprocal equations.

If α be a root of a reciprocal equation, $\frac{1}{\alpha}$ must also be a root, for it is a root of the transformed equation and the transformed equation is identical with the first equation, Hence the roots of a reciprocal equation occur in pairs

$$\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, \ldots$$

When the degree is odd one of its roots must be its own reciprocal.

$$\gamma = \frac{1}{\gamma}$$

i.e., $\gamma^2 = 1$.

i.e., $\gamma = \pm 1$.

If the coefficients have all like signs, then -1 is a root ; if the coefficients of the terms equidistant from the first and last have opposite signs, then $+1$ is a root. In either case the degree of an equation can be depressed by unity if we divide the equation by $x + 1$ or by $x - 1$. The depressed equation is always a reciprocal equation of even degree with like signs for its coefficients.

If the degree of a given reciprocal equation is even , say $n = 2m$ and if terms equidistant from the first and last have opposite signs, then

$$p_m = - p_m.$$

i.e., $p_m = 0$, so that in this type of reciprocal equations, the middle term is absent. Such an equation may be written as

$$x^{2m} - 1 + p_1 x(x^{2m-2} - 1) + \ldots.. 0 .$$

Dividing by $x^2 - 1$, this reduces to a reciprocal equation of like signs of even degree. Hence all reciprocal equations may be reduced to an even degree reciprocal equation with like sign, and so an even degree reciprocal equation with like signs is considered as the standard form of reciprocal equations.

A reciprocal equation of the standard form can always be depressed to another of half the dimensions.

It has been shown in the previous article that all reciprocal equations can be reduced to a standard form, in which the degree is even and the coefficients of terms equidistant from the beginning and the end are equal and have the same sign.

Let the standard reciprocal equation be

$$a_0 x^{2m} + a_1 x^{2m-1} + a_2 x^{2m-2} + \ldots.. a_m x^m + \ldots. + a_1 x + a_0 = 0.$$

Dividing by x^m and grouping the terms equally distant from the ends, we have

$$a_0\left(x^m + \frac{1}{x^m}\right) + a_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + \ldots + a_{m-1}\left(x + \frac{1}{x}\right) + a_m = 0$$

Let $x + \frac{1}{x} = z$ and $x^r + \frac{1}{x^r} = X_r$

We have the relation $X_{r+1} = z \cdot X_r - X_{r-1}$.

Giving r in succession the values 1, 2, 3, …

We have $X_2 = z X_1 - X_0 = z^2 - 2$

$X_3 = z X_2 - X_1 = z^3 - 3z$

$X_4 = z X_3 - X_2 = z^4 - 4z^2 + 2$

$X_5 = z X_4 - X_3 = z^5 - 5z^3 + 5z$

and so on. Substituting these values in the above equation. We get an equation of the m^{th} degree in *z*. To every root of the reduced equation in *z*, correspond two roots of the reciprocal equation. Thus if *k* be a root of the reduced equation, the quadratic $x + \frac{1}{x} = k$, i.e., $x^2 - kx + 1 = 0$ gives the two corresponding roots $\frac{k \pm \sqrt{k^2-4}}{2}$ of the given reciprocal equation.

Example 1. Find the roots of the equation $x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1 = 0$.

Solution.

This is a reciprocal equation of odd degree with like signs.

$\therefore$ (x+1) is a factor of $x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1$

The equation can be written as

$$x^5 + x^4 + 3x^4 + 3x^3 + 3x^2 + 3x + x + 1 = 0$$

i.e., $x^4(x+1) + 3x^3(x+1) + 3x(x+1) + 1(x+1) = 0$

i.e., $(x+1)(x^4 + 3x^3 + 3x + 1) = 0$.

$\therefore$ $x + 1 = 0$ or $x^4 + 3x^3 + 3x + 1 = 0$.

Dividing by x^2, we get $x^2 + 3x + \frac{3}{x} + \frac{1}{x^2} = 0$

$\left(x^2 + \frac{1}{x^2}\right) + 3\left(x + \frac{1}{x}\right) = 0$.

Put $x + \frac{1}{x} = z$. $\therefore$ $x^2 + \frac{1}{x^2} = z^2 - 2$

$\therefore$ $z^2 - 2 + 3z = 0$

$\therefore$ $z = \frac{-3 \pm \sqrt{17}}{2}$.

Hence $x + \frac{1}{x} = \frac{-3 \pm \sqrt{17}}{2}$

i.e., $2x^2 + \left(-3 + \sqrt{17}\right) x + 2 = 0$

or $2x^2 + \left(-3 - \sqrt{17}\right) x + 2 = 0$.

From these equations x can be found.

Example 2. Solve the equation $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$.

Solution.

This is a reciprocal equation of odd degree with unlike signs.

Hence $x - 1$ is a factor of the left- hand side.

The equation can be written as follows:

$6x^5 - 6x^4 + 5x^4 - 5x^3 - 38x^2 + 5x^2 - 5x + 6x - 6 = 0$

i.e., $6x^4 (x - 1) + 5x^3(x - 1) - 38x^2 (x - 1) + 5x(x - 1) + 6 (x - 1) = 0$

i.e., $(x - 1) (6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$

$\therefore$ $x - 1 = 0$ or $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$.

We have to solve the equation $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$.

Dividing by x^2, $6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$

i.e., $6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$.

Put $x + \frac{1}{x} = z$. $\therefore$ $x^2 + \frac{1}{x^2} = z^2 - 2$.

The equation becomes

$$6(z^2 - 2) + 5z - 38 = 0$$

i.e., $6z^2 + 5z - 50 = 0$

i.e., $(2z-5)(3z+10) = 0$.

$\therefore$ $x + \frac{1}{x} = \frac{5}{2}$ or $x + \frac{1}{x} = -\frac{10}{3}$

i.e., $2x^2 - 5x + 2 = 0$ or $3x^2 + 10x + 3 = 0$

i.e., $(2x - 1)(x - 2) = 0$ or $(3x + 1)(x + 3) = 0$

i.e., $x = \frac{1}{2}$ or 2 or $-\frac{1}{3}$ or -3.

$\therefore$ The roots of the equation are 1, $\frac{1}{2}$, 2, $-\frac{1}{3}$ and -3.

Example 3. Solve the equation $6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0$.

Solution.

There is no mid-term and this is a reciprocal equation of even degree with unlike signs. We can easily see that $x^2 - 1$ is a factor of the expression on left-hand side of the equation.

The equation can be written as

$$6(x^6 - 1) - 35x(x^4 - 1) + 56x^2(x^2 - 1) = 0$$

i.e., $6(x^2 - 1)(x^4 + x^2 + 1) - 35x(x^2 - 1) + (x^2 + 1)) + 56x^2(x^2 - 1) = 0$

i.e., $(x^2 - 1)(6x^4 - 35x^3 + 62x^2 - 35x + 6) = 0$

i.e., $x = 1$ or -1 or $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$.

Dividing by x^2, we get $6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2} = 0$.

$6\left(x^2 + \frac{1}{x^2}\right) - 35\left(x + \frac{1}{x}\right) + 62 = 0.$

Put $x + \frac{1}{x} = z$. $\therefore x^2 + \frac{1}{x^2} = z^2 - 2$.

$\therefore \quad 6(z^2 - 2) - 35z + 62 = 0$

i.e., $6z^2 - 35z - 50 = 0$

i.e., $(3z - 10)(2z - 5) = 0$

$z = \frac{10}{3}$ or $\frac{5}{2}$.

$\therefore \quad x + \frac{1}{x} = \frac{10}{3}$ or $x + \frac{1}{x} = \frac{5}{2}$

i.e., $3x^2 - 10x + 3 = 0$ or $2x^2 - 5x + 2 = 0$

i.e., $(x - 3)(3x - 1) = 0$ or $(x - 2)(2x - 1) = 0$

i.e., $x = 3$ or $\frac{1}{3}$ or 2 or $\frac{1}{2}$

$\therefore$ The roots of the equation are $1, -1, 3, \frac{1}{3}, 2$ and $\frac{1}{2}$.

Exercises

Solve the following equations:-

1. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.
2. $x^4 + 3x^3 - 3x - 1 = 0$.
3. $2x^6 - 9x^5 + 10x^4 - 3x^3 + 10x^2 - 9x + 2 = 0$.
4. $2x^5 + x^4 + x + 1 = 12x^2(x + 1)$.
5. $x^5 - 5x^3 + 5x^2 - 1 = 0$.

6. $x^6 + 2x^5 + 2x^4 - 2x^2 - 2x - 1 = 0$.

Answer : 1.$3 \pm \sqrt{8}$, $2 \pm \sqrt{3}$, 2. ± 1, $\frac{-3\pm\sqrt{5}}{2}$, 3. 2, $\frac{1}{2}, \frac{3\pm\sqrt{5}}{2}, \frac{-1\pm\sqrt{-3}}{2}$, 4.$-1, -2, -\frac{1}{2}, \frac{3\pm\sqrt{5}}{2}$,

5.1, 1, 1, $\frac{-3\pm\sqrt{5}}{2}$, 6. ± 1, $\frac{-1\pm i\sqrt{3}}{2}, \frac{-1\pm i\sqrt{3}}{2}$.

Transformation in general.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equations f (x) = 0, it is required to find an equation whose roots are

$\phi(\alpha_1), \phi(\alpha_2), \ldots, \phi(\alpha_n)$.

The relation between a root x of f (x) = 0 and a root y of the required equation is $y = \phi(x)$.

Now if x be eliminated between f (x) = 0 and $y = \phi(x)$, an equation in y is obtained which is the required equation.

By means of the relations between the roots and coefficients of an equation we can establish a relation between the corresponding roots given and the required equations. A few examples will illustrate the methods of procedure.

Example 1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r \equiv 0$, from the equation whose roots are $\alpha - \frac{1}{\beta\gamma}, \beta - \frac{1}{\gamma\alpha}, \gamma - \frac{1}{\alpha\beta}$.

Solution.

We have $\alpha - \frac{1}{\beta\gamma}$

$= \alpha - \frac{\alpha}{\alpha\beta\gamma}$

$= \alpha - \frac{\alpha}{-r}$ since $\alpha\beta\gamma = -r$

$= \alpha + \frac{\alpha}{r}$.

$\therefore$ $y = x + \frac{x}{r}$.

$\therefore$ The required equation is obtained by eliminating x between the equations

$$y = x + \frac{x}{r}. \qquad \ldots\ldots\ldots (1)$$

$$x^3 + px^2 + qx + r = 0 \qquad \ldots\ldots\ldots(2)$$

From (1) , we get $x = \frac{yr}{1+r}$

Substituting this value of x in the equation (2), we get

$$r^3y^3 + pr(1+r)\,y^2 + q(1+r)^2y + (1+r)^3 = 0.$$

Example 2. If a , b , c be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are $bc - a^2$, $ca - b^2$, $ab - c^2$ **.**

Solution.

We have $bc - a^2 = \frac{abc}{a} - a^2$

$= -\frac{r}{a} - a^2$ since $abc = -r$.

Hence the required equation is obtained by eliminating x between the equations

$$y = -\frac{r}{x} - x^2 \qquad \ldots\ldots\ldots (1)$$

and $x^3 + px^2 + qx + r = 0$ ……..(2)

From (1) , we get $x^3 + xy + r = 0$ …….(3)

Subtracting (3) from (2) , we get

$$px^2 + qx - xy = 0$$

i.e., $x(px + q - y) = 0$

i.e., $x = 0$ or $px + q - y = 0$.

x cannot be equal to zero.

$\therefore \quad px + q - y = 0.$

$$\therefore \quad x = \frac{y-q}{p}.$$

Substituting this value of x in equation (2) , we get

$$\left(\frac{y-q}{p}\right)^3 + p.\left(\frac{y-q}{p}\right)^2 + q.\left(\frac{y-q}{p}\right) + r = 0$$

i.e., $y^3 + (p^2 - 3q)\, y^2 + (3q^2 - p^2q)\, y + p^3r - q^3 = 0.$

Example 3. If α , β , γ are the roots of the equation $x^3 - 6x + 7 = 0$, from the equation whose roots are $\alpha^2 + 2\alpha + 3$, $\beta^2 + 2\beta + 3$, $\gamma^2 + 2\gamma + 3$.

Solution.

Here we have to eliminate x between the equations

$$x^3 - 6x + 7 = 0 \qquad \ldots\ldots\ldots (1)$$

and $\quad y = x^2 + 2x + 3$

i.e., $x^2 + 2x + (3 - y) = 0 \qquad \ldots\ldots\ldots(2)$

Multiplying (2) by x and subtracting (1) from it , we get

$$2x^2 + (9 - y)\, x - 7 = 0 \qquad \ldots\ldots\ldots\ldots(3)$$

From (2) and (3) , we get

$$\frac{x^2}{-14-(9-y)(3-y)} = \frac{x}{7+2(3-y)} = \frac{1}{(9-y)-4},$$

so that $(13 - 2y)^2 = (5 - y)(-y^2 + 12y - 41)$

i.e., $y^3 - 21y^2 + 153y - 374 = 0.$

Example 4. If α , β , γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of

$(\alpha^2 + 1)(\beta^2 + 1)(\gamma^2 + 1).$

Solution.

From the equation whose roots are

$\alpha^2+1, \beta^2+1, \gamma^2+1.$

For that, eliminate x between

$y = x^2+1$ ………(1)

and $x^3 + px^2 + qx + r = 0$ ………(2)

Equation (2) can be written as

$x(x^2 + q) = -(px^2 + r)$

i.e., $x(y - 1 + q) = -\{p(y-1) + r\}$ since from (1) $x^2 = y - 1$.

Squaring $x^2(y - 1 + q)^2 = \{p(y-1)+r\}^2$

i.e., $(y - 1)(y - 1 + q)^2 = \{p(y-1)+r\}^2$

i.e., $y^3 + y^2$ term + y term $-(q-1)^2 - (p-r)^2 = 0.$

The roots of the equation are $\alpha^2+1, \beta^2+1, \gamma^2+1.$

∴ Products of the roots

$(\alpha^2+1)(\beta^2+1)(\gamma^2+1) = (q-1)^2 + (p-r)^2.$

Example 5. If α is a root of $x^2(x+1)^2 - k(x-1)(2x^2 + x + 1) = 0$, prove that $\frac{\alpha+1}{\alpha-1}$ is also a root.

Solution.

From the equation whose roots are $\frac{\alpha+1}{\alpha-1}, \frac{\beta+1}{\beta-1}, \frac{\gamma+1}{\gamma-1}, \frac{\delta+1}{\delta-1}.$

For that, eliminate x between the equations

$y = \frac{x+1}{x-1}$ ……...(1)

and $x^2(x+1)^2 - k(x-1)(2x^2+x+1) = 0$ ……..(2)

From (1), we get $x = \frac{y+1}{y-1}$.

Substituting this value of x in (2), we get

$$\left(\frac{y+1}{y-1}\right)^2 \left\{\frac{y+1}{y-1}+1\right\}^2 + k\left(\frac{y+1}{y-1}-1\right).\left\{2.\left(\frac{y+1}{y-1}\right)^2 + \frac{y+1}{y-1}+1\right\} = 0$$

i.e., $(y+1)^2(2y)^2 - k \cdot 2 \cdot (y-1) \cdot \{2(y+1)^2 + y^2 - 1 + (y-1)^2\} = 0$

i.e., $4y^2(y+1)^2 - k \cdot 2 \cdot (y-1)(4y^2+2y+2) = 0$

i.e., $y^2(y+1)^2 - k \cdot (y-1)(2y^2+y+1) = 0$.

We get the same equation as the original equation.

$\therefore$ $\frac{\alpha+1}{\alpha-1}$ is a root of $x^2(x+1)^2 - k \cdot (x-1)(2x^2+x+1) = 0$.

Example 6. Find the equation whose roots are the squares of the differences of the roots of the equation $x^3+px+q=0$ (p and q being real). Hence deduce the condition that all the roots of the cubic shall be real.

Solution.

Let α, β, γ be the roots of the equation $x^3+px+q=0$.

We have to form the equation whose roots are $(\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$.

$(\beta-\gamma)^2 = \beta^2+\gamma^2-2\beta\gamma$

$= \alpha^2+\beta^2+\gamma^2-\alpha^2-\frac{2\alpha\beta\gamma}{\alpha}$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) - \alpha^2 - \frac{2\alpha\beta\gamma}{\alpha}$.

Here we have $\alpha+\beta+\gamma = 0$, $\alpha\beta+\beta\gamma+\gamma\alpha = p$, $\alpha\beta\gamma = -q$.

$\therefore$ $(\beta-\gamma)^2 = -2p - \alpha^2 + \frac{2q}{\alpha}$.

Hence to get the transformed equation eliminate x between the equations

$$y = -2p - x^2 + \frac{2q}{x} \qquad \text{.........(1)}$$

and $x^3 + px + q = 0$ ....(2)

(1) can be written as

$$x^3 + (y + 2p)\,x - 2q = 0 \qquad \text{........(3)}$$

Subtracting (2) from (3) , we get (y + p) x – 3q = 0.

$$\therefore\ x = \frac{3q}{y+p}.$$

Substituting this value of x in (1), we get

$$\left(\frac{3q}{y+p}\right)^3 + p\left(\frac{3q}{y+p}\right) + q = 0.$$

Simplifying $y^3 + 6py^2 + 9p^2 y + 4p^2 + 27q^2 = 0.$

$$\therefore\ (\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2 = -(4p^2 + 27q^2).$$

If α ,β ,γ are real, then α − β , β − γ , γ − α are real and may be positive or negative .

$\therefore$ $(\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$ are all positive.

Hence (1) $(\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$ is +ve

i.e., $4p^2 + 27q^2$ is –ve.

(2) $(\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$ is +ve

i.e., $-6p$ is +ve

i.e., p is –ve.

$4p^2 + 27q^2$ is negative implies that p is –ve.

$\therefore$ The condition for the roots of the equation to be real is $4p^2 + 27q^2$ is negative.

Cubic equation.

1. Let the cubic equation be $x^3 + ax + b = 0$.

Method 1. The equation can be written as $x^3 = -ax - b$.

The x-coordinates of the intersection of the curves $y = x^3$ and $y = -ax - b$ will be give the roots of the equation.

$y = x^3$ curve has a point of inflection at the origin.

Method 2. Multiply the equation by x.

We get $x^4 + ax^2 + bx = 0$

i.e., $(x^2)^2 + x^2 + (a - 1)x^2 + bx = 0$.

We can easily see that the roots of the equation are the x- coordinates of the points of intersection of the parabola $y = x^2$ and the circle $y^2 + y + (a - 1)y + bx = 0$.

Here the origin is to be excluded since we have multiplied the equation by x.

2. If the cubic equation is $ax^3 + bx^2 + cx + d = 0$, we can diminish the roots of the cubic by h and get an equation without the x^2 term. One of the above two methods can be adopted to get the roots. The equation can be written as

$$ax^3 = -bx^2 - cx - d$$

$$\text{i.e., } x^3 = -\frac{b}{a}x^2 - \frac{c}{a}x - \frac{d}{a}.$$

The roots are the x-coordinates of the intersection of the curves

$$y = x^3$$

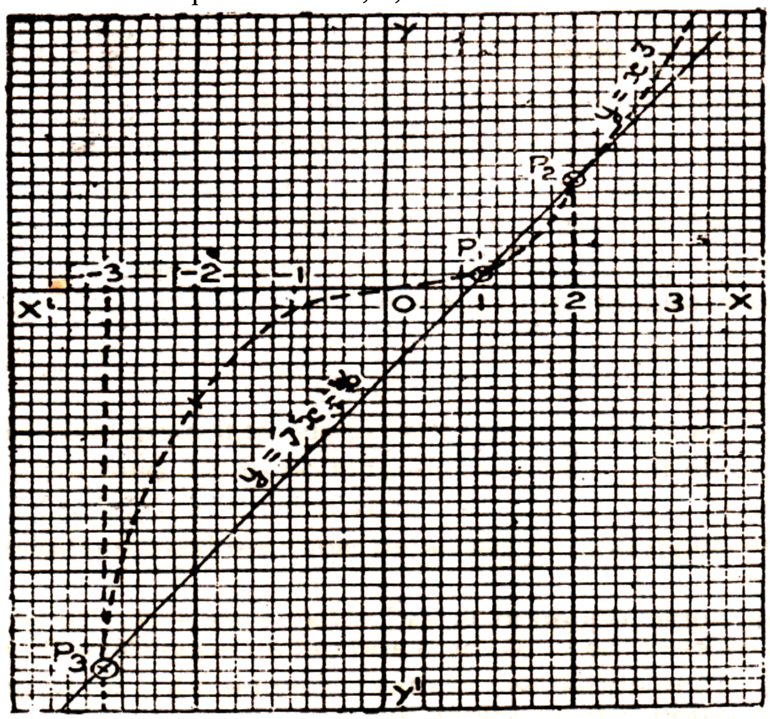
$$\text{and } \quad y = -\frac{b}{a}x^2 - \frac{c}{a}x - \frac{d}{a}.$$

Example 1. Find graphically all the roots of $x^3 - 7x + 6 = 0$.

Solution.

Method 1. The equation can be written as $x^3 = 7x - 6$. The x-coordinates of the points of intersection of the curve $y = x^3$ and the straight line $y = 7x - 6$ will give the roots of the equation. The line $y = 7x - 6$ intersects the curve in three real points and x-coordinates of the points are 1, 2, -3.

∴ The roots of the equations are 1, 2, -3.



Method 2. Multiply the equation by x .

We get $x^4 - 7x^2 + 6x = 0$

i.e., $(x^2)^2 + x^2 - 8x^2 + 6x = 0$.

The roots are the x- coordinates of the intersection of the curves

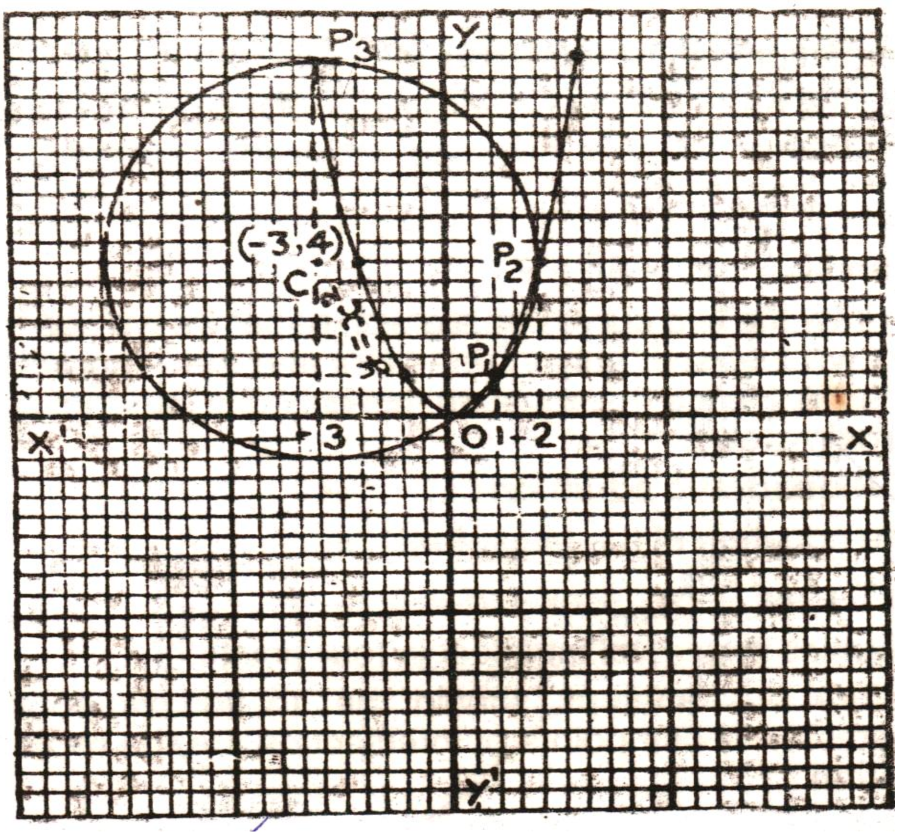
$$y = x^2$$

and $y^2 + y - 8y + 6x = 0$.

The first curve is a parabola and the second is a circle whose centre is (– 3 , 4) and the radius 5.

By drawing the curves, we can see that the curves intersect at the point whose x-coordinates are 1 , 2 , and – 3.

∴ The roots of the equation are 1 , 2 , and – 3.



Example 2. Show that the equation $x^3 – 3x^2 + 3x – 7 = 0$ has only one real root. Find the root graphically to the first decimal place.

Solution.

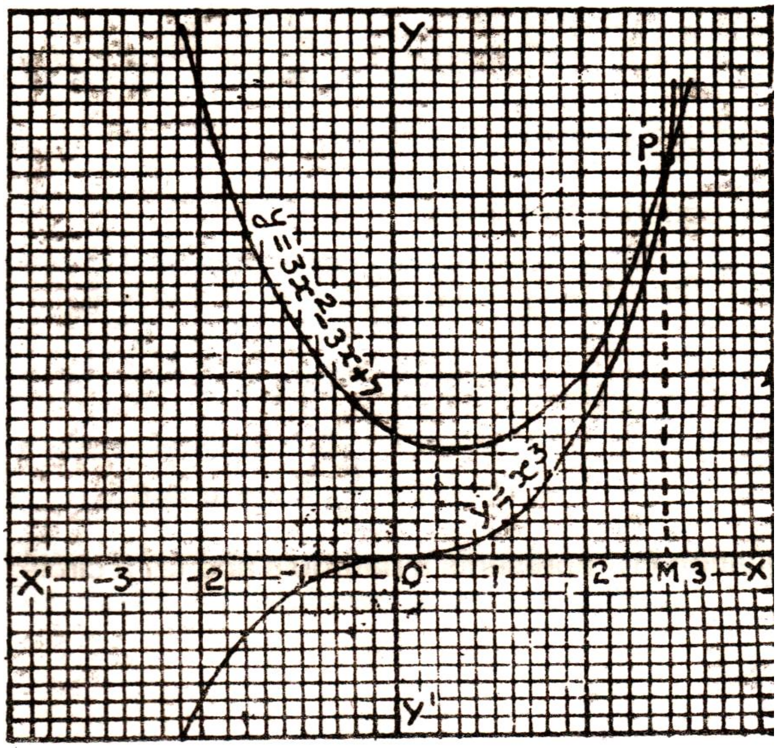
The equation can be written as $x^3 = 3x^2 – 3x + 7$.

Hence the root are the x-coordinates of the points intersection of the curves

$$y = x^3$$

and $y = 3x^2 - 3x + 7$.

If we draw the two curves on a graph paper we will see that the two curves will intersect only in one point and the x-coordinate of that point is 2.8. Hence the equation has only one real root and that is 2.8 approximately.



Bi quadratic equations.

Let the equation of the bi quadratic be $x^4 + ax^3 + bx^2 + cx + d = 0$.

Two conics in general intersect in four points.

Therefore our attempt should be to find two conics, the x- coordinates of whose points of intersection are the roots of the given equation.

The equation can be written as

$$(x^2 + \frac{a}{2}x)^2 - \frac{a^2}{4}x^2 + bx^2 + cx + d = 0.$$

i.e., $(x^2 + \frac{a}{2}x)^2 + x^2 + (b - 1 - \frac{a^2}{4})x^2 + cx + d = 0.$

Let $y = x^2 + \frac{a}{2}x$ ……….(1)

Then the equation becomes

$$y^2 + x^2 + (b - 1 + \frac{a^2}{4})\,(y - \frac{a}{2}x) + cx + d = 0$$

i.e., $x^2 + y^2 - \frac{a}{2}(b - 1 - \frac{a^2}{4} - \frac{2c}{a})\,x + (b - 1 + \frac{a^2}{4})\,y + d = 0$ …….(2)

The equation (1) represents a parabola and (2) a circle.

Trace the curves on a graph paper and the x- coordinates of the points of intersection are the roots of the given equation .

Example. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 19 = 0$ graphically.

Solution.

The equation can be written as $(x^2 - x)^2 + 3x^2 + 6x - 19 = 0$

Let $y = x^2 - x$ ………..(1)

Then the equation becomes $y^2 + x^2 + 2x^2 + 6x - 19 = 0$

i.e., $x^2 + y^2 + 2(y+x) + 6x - 19 = 0$

i.e., $x^2 + y^2 + 8x + 2y - 19 = 0$

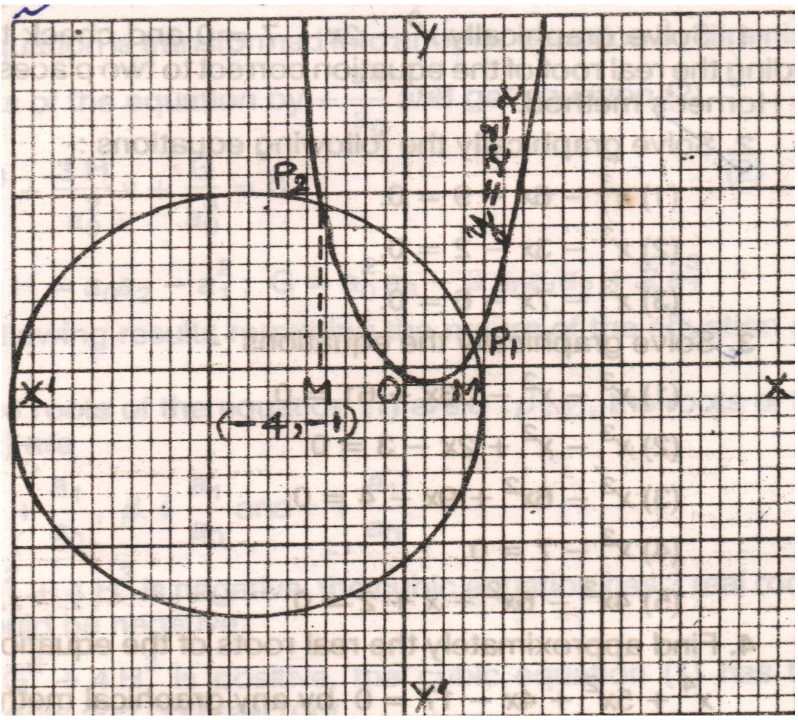
i.e., $(x + 4)^2 + (y + 1)^2 = 36$

i.e., $(x + 4)^2 + (y + 1)^2 = 6^2$ ………..(2)

Trace the curves (1) and (2) on a graph paper.

The curves intersect only in two real points.

Therefore the given equation has only two real roots and they are approximately 1.6 and 1.7.



Exercises

1. Solve graphically the following equations:-

   (1) $x^3 - 6x - 9 = 0$

   (2) $x^3 - 3x - 2 = 0$

   (3) $x^3 - 7x - 6 = 0$

2. Solve graphically the following equations:-

   (1) $x^3 - x^2 - 33x + 61 = 0$

   (2) $x^3 - x^2 + 2x - 3 = 0$

   (3) $x^3 - 6x^2 + 9x - 4 = 0$

   (4) $x^3 - 7 = 0$

   (5) $4x^3 - 6x^2 + x + 2 = 0$

3. Solve graphically by using $y = x^2$ and a circle or otherwise $3x^4 - x^2 + 3x - 4 = 0$

Cardon's method.

Let the equation be $x^3 + px + q = 0$. ……….. (1)

Let x be u + v. Substituting this value of x in equation (1), we get

$(u + v)^3 + p(u + v) + q = 0.$

i.e., $u^3 + v^3 + 3uv(u + v) + p(u + v) + q = 0$

i.e., $u^3 + v^3 + q + (u + v)(3uv + p) = 0.$

Choose u and v such that $3uv + p = 0$.

Then the equation reduces to

$$u^3 + v^3 + q = 0 \qquad \ldots\ldots\ldots\ldots(2)$$

with the condition $3uv + p = 0$ ....(3)

Eliminate u from (2) and (3) , we get

$$\left(-\frac{p}{3v}\right)^3 + v^3 + q = 0$$

$$\text{i.e., } v^6 + qv^3 - \frac{p^3}{27} = 0 \qquad \ldots\ldots\ldots\ldots(4)$$

Similarly eliminate v from (2) and (3) , we get

$$u^6 + qu^3 - \frac{p^3}{27} = 0 \qquad \ldots\ldots\ldots\ldots(5)$$

From (4) and (5) relations, we get that

u^3 and v^3 are the roots of the equation

$$t^2 + qt - \frac{p^3}{27} = 0 \qquad \ldots\ldots\ldots\ldots(6)$$

u^3 and v^3 can be determined from this equation

$$u^3 = -\frac{q}{2} + \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{\frac{1}{2}}, \quad v^3 = -\frac{q}{2} - \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{\frac{1}{2}}$$

Roots of equation (6) are real only when $\frac{q^4}{4} + \frac{p^3}{27} \geq 0.$ In that case two roots of equation (1) are imaginary and one root real or two of the roots of the equation (1) are real.

Let $\frac{q^4}{4} + \frac{p^3}{27}$ i.e., $4p^3 + 27q^2$ is positive.

Then u^3 and v^3 are real and let $u^3 = m^3$ and $v^3 = n^3$. Here we obtain 3 values of u viz.,

m , ωm , ω^2m and 3 values of v viz., n , ωn , ω^2n , where ω and ω^2 are the cube roots of unity.

Hence we get 9 combinations for u + v. Out of the nine combinations, the following 3 combinations values are only valid for u + v since.

$u^3v^3 = -\frac{p^3}{27}$ i.e., $uv = -\frac{p}{3}$.

$m + n$; $m\omega + n\omega^2$ and $m\omega^2 + n\omega$.

Hence they are the roots of the given equation (1). The solution of the cubic equation depends on the roots of the equation (6).

The roots of the equation (6) are imaginary if $q^2 + \frac{4p^3}{27} < 0$. In that case both u^3 and v^3 are imaginary and hence u and v are the cubic roots of imaginary quantities. This has no arithmetical meaning. Hence Cardon's method is not useful. So before trying to solve a cubical equation, find the nature of its roots. If all the three roots are real we can not use Cardon's method to get arithmetic values for the roots.

Example 1. Solve the equation $x^3 - 6x - 9 = 0$.

Solution.

Here p = – 6 and q = – 9

$$4p^3 + 27q^2 = 4(-6)^3 + 27(-9)^2 = 1323 > 0.$$

Hence the equation has no real root and two imaginary roots and so Cardon's method is applicable.

$\therefore$ x = u + v where u^3 and v^3 are the roots of the equation

$$t^2 + qt - \frac{p^3}{27} = 0$$

i.e., $t^2 - 9t + 8 = 0$

i.e., $(t - 8)(t - 1) = 0$

$\therefore$ $u^3 = 8$ and $v^3 = 1$.

Hence 2+1 i.e., 3 is one of the roots of the equation. The other roots are $2\omega + \omega^2$ and $2\omega^2 + \omega$. Or since 3 is one of the roots of the equation, dividing the given equation by x – 3 , we get the other rots of the given equation. They are the roots of the equation $x^3 + 3x + 3 = 0$. Hence the given equation has the 3 roots

$$3, \frac{-3+i\sqrt{3}}{2}, \quad \frac{-3-i\sqrt{3}}{2}$$

These are the same as 3, $2\omega + \omega^2$ and $2\omega^2 + \omega$.

Example 2. Solve the equation $x^3 - 9x^2 + 108 = 0$.

Solution.

Transform this equation into one without the second term, i.e., the term without x^2 term. This can be done by decreasing the roots by 3.That equation is $x^3 - 27x + 54 = 0$.

If α, β, γ are the roots of the equation (1), the roots of the given equation are α + 3, β + 3 and γ + 3.

Here u^3 and v^3 are the roots of the equation.

$t^2 + qt - \frac{p^3}{27} = 0$ where q = 54, p = $-$ 27.

i.e., $t^2 + 54t + (27)^2 = 0$

i.e., $(t + 27)^2 = 0$

Hence two of the roots of the equation are equal

$\therefore$ $u^3 = -27$ and $v^3 = -27$.

Hence u = $-$ 3 and v = $-$ 3.

∴ The roots of the equation (1) are

$$-6, -3\omega \ -3\omega^2 \text{ and } -3\omega^2 - 3\omega$$

Since ω and ω^2 are the cubic roots of the unity

$$1 + \omega + \omega^2 = 0.$$

Hence these roots are $-6, 3, 3$.

∴ The roots of the given equation are -3, 6 and 6.

Solution of bi quadratic equations.

Of the several methods of solution of a bi quadratic equation, the simplest is due to Ferrari. The method is illustrated below.

Let the equation be

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

Express the left side of the equation as the difference of squares of a quadratic function and a linear function.

The equation can be written as

$$(x^2 + \frac{p}{2}x)^2 - (q - \frac{p^2}{4})x^2 + rx + s = 0.$$

The equation can be expressed as

$$(x^2 + \frac{p}{2}x + \lambda)^2 - \left\{\left(\frac{p^2}{4} - q + 2\lambda\right)x^2 + (\lambda p - r)x + \lambda^2 - s\right\} = 0$$

Which is of the form

$$(x^2 + \frac{p}{2}x + \lambda)^2 - (\alpha x + \beta)^2 = 0$$

Where $\alpha^2 = \frac{p^2}{4} - q + 2\lambda$, $2\alpha\beta = \lambda p - r$, $\beta^2 = \lambda^2 - s$.

Eliminating α and β from these equations, we get

$$4\left(\frac{p^2}{4} - q + 2\lambda\right)(\lambda^2 - s) = (\lambda p - r)^2$$

At least the root of this cubic equation in λ is real. From the real root of this equation, α and β can be determined.

Hence the given equation can be factorized into

$$x^2 + \frac{p}{2}x + \lambda \pm (\alpha x + \beta) = 0.$$

Solving the two quadratic equations, all the four roots of the bi quadratic can be determined. Hence the solution of the bi quadratic equation depends on the solution of a cubic equation, which can be solved by Cardon's Method or by trial and error method which is explained in the article Newton's method of divisors.

Example 1. Solve the equation $4x^4 + 4x^3 - 7x^2 - 4x - 12 = 0$

This equation can be written as

$$(2x^2 + x)^2 - ax^2 - 4x - 12 = 0$$

i.e., $(2x^2 + x + \lambda)^2 - \{(4\lambda + 8)x^2 + (2\lambda + 4)x + \lambda^2 + 12\} = 0$

i.e., $(2x^2 + x + \lambda)^2 - (\alpha x + \beta)^2 = 0$

Where $\alpha^2 = 4(\lambda + 2)$, $2\alpha\beta = 2(\lambda + 2)$, $\beta^2 = \lambda^2 + 12$

Eliminating α and β from these relations, we get

$$16(\lambda + 2)(\lambda^2 + 12) = 4(\lambda + 2)^2$$

Which reduces to

$$(\lambda + 2)(4\lambda^2 - \lambda + 46) = 0.$$

The only real root of the equation is – 2.

Hence the given equation reduces to

$$(2x^2 + x - 2)^2 - (4)^2 = 0$$

i.e., $(2x^2 + x + 2)(2x^2 + x - 6) = 0$

The roots of $2x^2 + x + 2 = 0$ are $\frac{-1 \pm i\sqrt{15}}{4}$

The roots of $2x^2 + x - 6 = 0$ are -2 and $\frac{3}{2}$.

∴ The roots of the given biquadratic equation are

$-2, \frac{3}{2}$ and $\frac{-1 \pm i\sqrt{15}}{4}$.

Example 2. Solve the equation $x^4 - 4x^3 - 10x^2 + 64x + 40 = 0$

This equation can be written as

$$(x^2 - x)^2 - 14x^2 + 64x + 40 = 0$$

and hence as

$$(x^2 - 2x + \lambda)^2 - \{(2\lambda+14)\,x^2 - (4\lambda+64)\,x + \lambda^2 - 40\} = 0$$

i.e., $(x^2 - 2x + \lambda)^2 - (\alpha x + \beta)^2 = 0$

Where $\alpha^2 = 2(\lambda+7)$, $2\alpha\beta = -4(\lambda+16)$, $\beta^2 = \lambda^2 - 40$.

Eliminating α and β from these relations, we get

$$8(\lambda + 7)(\lambda^2 - 40) = 16(\lambda + 16)^2$$

On simplification, this equation reduces to

$$\lambda^3 + 5\lambda^2 - 104\lambda - 792 = 0$$

$792 = 2^3 3^3 (11)$. On trial we find that 11 satisfies the equation.

Dividing by $\lambda - 11$, we get $\lambda^2 + 16\lambda + 72 = 0$ which gives imaginary roots.

When $\lambda = 11$, $\alpha = \pm 6$, $\beta = \pm 9$, $\alpha\beta = -54$.

∴ $\alpha = 6, \beta = -9$ or $\alpha = -6, \beta = 9$.

Both pairs of the values will lead to the same factorization of the expression on the left side. Hence the equation reduces to

$(x^2 - 2x + 11)^2 - (6x - 9)^2 = 0$

i.e., $(x^2 + 4x + 2)(x^2 - 8x + 20) = 0$

Hence the roots of the given equation are

$-2 \pm \sqrt{2}$ and $4 \pm 2i$.

Example 3. Solve the equation $2x^4 + 6x^3 - 3x^2 + 2 = 0$.

Solution.

Transform this equation into another whose roots are twice the roots of the given equation. The transformed equation is

$2x^4 + 6(2)x^3 - 3(2^2)x^2 + 2(2^4) = 0$

Which reduces to

$x^4 + 6x^3 - 6x^2 + 16 = 0$ …………..(1)

This equation can be written as

$(x^2 + 3x + \lambda)^2 - \{(2\lambda+15)x^2 + 6\lambda x + \lambda^2 - 16\} = 0$

The equation can be expressed as

$(x^2 + 3x + \lambda)^2 - (\alpha x + \beta)^2 = 0$ where

$\alpha^2 = 2\lambda + 15$, $2\alpha\beta = 6\lambda$, $\beta^2 = \lambda^2 - 16$.

Eliminating α and β from these relations, we get

$4(2\lambda + 15)(\lambda^2 - 16) = 36\lambda^2$

Simplifying we get $2\lambda^3 + 6\lambda^2 - 32\lambda - 240 = 0$ ……….(2)

$240 = 2^4(3)(5)$, by trial we see that $\lambda = 5$ is a root of (2).

Hence $\alpha = \pm 5$, $\beta = \pm 3$, $\alpha\beta = 15$.

$\therefore$ $\alpha = 5$, $\beta = 3$ or $\alpha = -5$, $\beta = -3$.

Hence for both pairs of values (1) reduces to

$$(x^2 + 3x + 5)^2 - (5x + 3)^2 = 0.$$

i.e., $(x^2 + 8x + 8)(x^2 - 2x + 2) = 0$

The root of this equation are $-4 \pm 2\sqrt{2}$ and $1 \pm i$. Hence the roots of the given equation are $-2 \pm \sqrt{2}$ and $\frac{1 \pm i}{2}$.

Exercises

Solve the equations

1. $x^3 + 3x^2 + 6x + 4 = 0$
2. $x^3 - x^2 - 16x + 20 = 0$
3. $x^3 + 6x^2 + 9x + 4 = 0$
4. $3x^4 - 10x^3 + 6x^2 - 10x + 3 = 0.$
5. $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0.$

Answers : 1. –1, $-1 \pm i\sqrt{3}$, 2. 2, 2, –5, 3. –4, –1, –1, 4. ± i, 3, $\frac{1}{3}$, 5. –1, –3, 1 ± 2i.

Solution of numerical equation

An equation such as $3x^3 - 2x^2 - 5x + 7 = 0$, where coefficient are numbers are called numerical equation. Such an equation may have real and imaginary roots. Among the real roots, some roots may be commensurable and some incommensurable. We shall give below some methods to determine the commensurable and approximations to incommensurable roots of a numerical equation.

A rational fraction cannot be a root of an equation with integral coefficients, the coefficient of x^n being unity

If possible let $\frac{a}{b}$ (a fraction in its lowest terms) be a root of the equation $x^n + p_1x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$, where $p_1, p_2, p_3, \ldots, p_n$ are integers.

Therefore $\left(\frac{a}{b}\right)^n + p_1 \left(\frac{a}{b}\right)^{n-1} + p_2 \left(\frac{a}{b}\right)^{n-2} + \ldots + p_n = 0.$

Multiplying throughout by b^{n-1}, we get

$-\frac{a^n}{b} = p_1a^{n-1} + p_2\, ba^{n-2} + p_3\, b^2a^{n-3} + \ldots + p_nb^{n-1}$ ….(1)

a is not divisible by b.

Therefore $\frac{a^n}{b}$ = a fraction.

But each term on the right side of (1) is an integer. We have therefore a fraction equal to an integer which is impossible. Hence $\frac{a}{b}$ cannot be a root of the equation. So the real roots of the equation are either integer or incommensurable roots.

Integral roots

Since p_n is numerically equal to the product of all the roots, it is evident that integral roots are the exact divisors of p_n. Hence to find the integral roots of an equation we have to find the factors of p_n which satisfy the equation. If the coefficient of x^n is not unity but p_0 then transform the equation into another whose roots are those of the given equation multiplied by p_0. In the new equation p_0 will be a common factor in all the coefficients of the terms. We can divide the equation by p_0 and get an equation with the coefficient of the first term unity.

Example 1. Solve the equation $x^4 + 2x^3 - x - 2 = 0$. The integral roots must be found among the values ± 1, ± 2 which are the factors of – 2. By Descartes' rule of signs. It can have at the most one positive root.

Solution.

Substituting these value in the expression on the left side, we can see that 1, – 2 are the roots of the equation.

We can easily see that

$x^4 + 2x^3 - x - 2 = (x - 1)(x + 2)(x^2 + x + 1)$.

∴The other roots of the equation are $\frac{-1 \pm \sqrt{-3}}{2}$

Example 2. Find the rational root of $2x^3 - x^2 - x - 3 = 0$ and hence complete the solution of the equation.

Solution.

Multiply the roots of the equation by 2.

$2x^3 - 2x^2 - 4x - 24 = 0$.

$x^3 - x^2 - 2x - 12 = 0$.

The integral roots must be found among the values $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Substituting these values in the equations, we find + 3 is the only root

$\frac{3}{2}$ is the only rational root of the original equation.

We can easily see that the other roots are $\frac{-1 \pm \sqrt{-3}}{2}$.

If we can find limits between which the real roots of an equation lie it is possible to limit the number of trials. We shall give below some elementary methods to determine such limits. One way of finding the upper limit is to group the term of the equations in such a way that each group is separately positive. Consider for example the following equations:

(1) $2x^3 - 5x^2 + x + 10 = 0$.

This may be written in the form

$x^2(2x - 5) + (x + 10) = 0$.

If $x > 3$, each one of the group is positive. Thus the upper limit of the real roots may be taken as 3.

(2) $3x^4 + 6x^3 + 12x^2 - 4x - 10 = 0$.

i.e., $3x^4 + (6x^3 - 4x) + 12x^2 - 10 = 0$.

i.e., $3x^4 + 2x\,(3x^2 - 2) + 2(6x^2 - 5) = 0$.

Each one of the group is positive if $x > 1$. The upper limit may be taken as 1.

(3) $5x^5 - 7x^4 - 10x^3 - 23x^2 - 90x - 417 = 0$.

Distributing the higher power of x among the negative terms, the equation may be written as

$x^5 - 7x^4 + x^5 - 10x^3 + x^5 - 23x^2 + x^5 - 90x + x^5 - 417 = 0$.

i.e., $x^4(x - 7) + x^3\,(x^2 - 10) + x^2(\,x^3 - 23) + x\,(x^4 - 90) + (x^5 - 417) = 0$.

If $x > 7$, each group is positive. Hence the upper limit may be taken as 7.

(4) $x^4 - x^3 - 2x^2 - 4x - 24 = 0$.

Multiplying the equation by 4 and distributing the highest powers among the negative terms, we get

$(x^4 - 4x^3) + (x^4 - 8x^2) + (x^4 - 16x) + (x^4 - 96) = 0$.

$x^3(x - 4) + x^2(x^2 - 8) + x(x^3 - 16) + (x^4 - 96) = 0$.

Here the upper limit is 4.

To find the lower limits of the real roots it is enough to find the lower limits of the negative roots of the equation. The negative roots of $f(+\,x) = 0$ are the positive roots of f(- x) = 0. Hence the lower limit of the negative roots of f(x) = 0 is the upper limit (with the sign changed) of the positive roots of the equation f(- x) = 0.

If the numerically greatest negative coefficient in the equation f(x) = $x^n + p_1x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ is equal to – p, then p + 1 is an upper limit to the positive roots.

f (x) > 0 if $x^n > p(x^{n-1} + x^{n-2} + \ldots + 1)$

i.e., if $> p\,\frac{x^n - 1}{x - 1}$

i.e., if $> p\,\frac{x^n}{x - 1}$

i.e., if $x^n\left(1 - \frac{p}{x-1}\right) > 0$

i.e., if $\left(1 - \frac{p}{x-1}\right) > 0$

i.e., x – 1 > p

i.e., x > p +1.

Hence according to this rule the upper limits in the previous article are 6, 11, 18, 25.

If α is a root of the equation

f(x) = $x^n + p_1x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ then x – α is a factor of f (x).

let the quotient when f (x) is divided by x – α be

$x^{n-1} + b_1x^{n-2} + \ldots + b^{n-1}$.

Hence we have

$f(x) \equiv (x - \alpha)(x^{n-1} + b_1x^{n-2} + \ldots + b^{n-1})$

if we put x = k in the identity, we have

$f(k) = (k - \alpha)(k^{n-1} + b_1k^{n-2} + \ldots + b^{n-1})$.

Therefore k – α is a factor of f (k).

In particular if k = 1, or – 1, f(1) is divisible by 1 – α and f (– 1) is divisible by – 1 – α, i.e., f(1) is divisible by α – 1 and f (– 1) by α + 1. Before testing any divisor α for a root, calculate f (1) and f (– 1) such of the divisor decreased by 1 which fail to divide exactly f (1) and the divisors increased by 1 which fail to divide f (– 1) are to be rejected.

Newton's method of divisors.

Let the given equation be

$f(x) = p_0x^n + p_1x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ where p_0, p_1, p_2, …, p_n are integers.

If α is a rational root of the equation, then α is a factor of p_n and f (x) is exactly divisible by x – α

Let the equation when f (x) is divided by x – α

$b_0x^{n-1} + b_1x^{n-2} + \ldots + b^{n-1}$.

Here $b_0, b_1, b_2, \ldots, b^{n-1}$ are integers

$\therefore\ p_0x^n + p_1x^{n-1} + p_2 x^{n-2} + \ldots + p_n = (x - \alpha)(b_0x^{n-1} + b_1x^{n-2} + \ldots + b^{n-1})$

Equating the coefficient of like powers on both sides, we have

$p_0 = b_0$

$p_1 = b_1 - \alpha b_0$ or $p_1 - b_1 = -\alpha b_0$

$p_2 = b_2 - \alpha b_1$ or $p_2 - b_2 = -\alpha b_1$

……………………………………

……………………………………

$p_r = b_r - \alpha b_{r-1}$ or $p_r - b_r = -\alpha b_{r-1}$

…………………………………….

$p_{n-1} = b_{n-1} - \alpha b_{n-2}$ or $p_{n-1} - b_{n-1} = -\alpha b_{n-2}$.

$p_n = -\alpha b_{n-1}$

p_n is divisible by α and the quotient is $-b_{n-1}$

p_n,	p_{n-1},	p_{n-2},	…	p_2,	p_1,	p_0
	$-b_{n-1}$,	$-b_{n-2}$,	……	$-b_2$,	$-b_1$,	$-b_0$
	$-\alpha b_{n-2}$	$-\alpha b_{n-3}$	……	$-\alpha b_1$	$-\alpha b_0$	0

In the first line the successive coefficients p_0, p_1, p_2, …, p_n in the reverse order of their occurrence are written and the quotient $-b_{n-1}$ when p_n is divided by α is written below p_{n-1} and added, we get $-\alpha b_{n-2}$. If this is divided, by α and the quotient– b_{n-2} is written below p_{n-2} and added, we get $-\alpha b_{n-3}$. If we continue this process, in the end we get zero since $p_0 = b_0$. Since $b_0, b_1, b_2, \ldots, b_{n-1}$ are integers, if at any stage the quotient we get is a fraction, we can at once infer that α is not a root of f (x) = 0. Also the last quotient b_0 must be equal to p_0.

Example 1. Solve the equation $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0$ by finding the rational roots

Solution.

The equation can be grouped as follows:

$x^4 - 5x^3 + 3x^3 - 13x^2 + 38x - 24 = 0$

i.e., $x^3(x - 5) + x^2 (3x - 13) + 2(19x - 12) = 0$

Hence the upper limit of the real root is 5.

Changing x into – x, we get $x^4 + 2x^3 - 13x^2 - 38x - 24 = 0$

The term of the equation can be grouped as follows:

$x^4 - 13x^2 + x^3 - 38x + x3 - 24 = 0.$

i.e., $x^2(x^2 - 13) + x\,(x^2 - 38) + (x^3 - 24) = 0$

if x = 7, each group is positive.

Therefore the lower limit of the negative root is – 7,

Hence the real roots of the equation lies between – 7 and 5.

The divisors of 24 (other than ± 1) are ± 2, ± 3, ± 4, ± 6, ± 8, ± 12, ± 24.

Since the real roots lies between – 7 and 5 it is enough to test for roots the divisors

± 1, ± 2, ± 3, ± 4, – 6.

Here $f(x) = x^4 - 2x^3 - 13x^2 + 38x - 24$.

Hence f (1) = 0 and f (– 1) = – 72

∴ 1 is a root of the equation and (– 1) is not a root of it.

If α is a root of f (x) = 0, then α + 1 is a factor of f (–1).

If 1 is added to the divisors to be tested, i.e., ± 2, ± 3, ± 4, – 6 , we get 3, 4, 5, – 1, – 2, –3, – 5,

5 and – 5 are not factors of – 72.

Hence the divisor 4 and – 6 are to be rejected.

We can apply Newton's method of divisors for obtaining the rational roots.

– 2	–24	38	– 13	– 2	+ 1
		12	–25	19	
		50	– 38	17	

The trial divisor – 2 has to be rejected since it does not divide 17 exactly

2	–24	38	– 13	–2	+ 1	
		–12	13	0	– 1	
		26	0	– 2	0	

Hence 2 is root of the equation. When f (x) is divided by x – 2, we get the quotient $x^3 - 13x + 12 = 0$. We shall test the other divisors ± 3, – 4 on the equation.

–3	12	– 13	0	1
		– 4		
		– 17		

Therefore – 3 is not a root.

3	12	– 13	0	1
		4	– 3	– 1
		–9	– 3	0

Therefore 3 is a root of the equation

–4	– 4	3	1
		1	– 1
		4	0

Therefore – 4 is a root of the equation and hence the roots of the equation are 1, 2, 3 and – 4.

Example 2. Find all the rational roots of the equation $4x^3 + 20x^2 - 23x + 6 = 0$ ……(1)

Solution.

Multiply the roots of the equation m.

Then the transformed equation becomes

$4x^3 + 20mx^2 - 23m^2x + 6m^3 = 0$

If we take m = 2 then 4 will become the common factor of all the terms of the equation.

In that case the equation becomes $x^3 + 10x^2 - 23x + 12 = 0$ ………….(2)

Find the rational roots of the equation. These rational roots of the equation will be twice the roots of the original equation. The transformed equation can be written as

$x^3 + x(10x - 23) + 12 = 0$

When x = 3, the expression on the left side is positive. Hence 3 is the upper limit of the real roots.

Changing x into – x, the equation becomes

$-\ x^3 + 10x^2 + 23x + 12 = 0$

i.e., $x^3 - 10x^2 - 23x - 12 = 0$

i.e., $3x^3 - 30x^2 - 69x - 36 = 0$

i.e., $(x^3 - 30x^2) + (x^3 - 69x) + (x^3 - 36) = 0$

i.e., $x^2 (x - 30) + x(x^2 - 69) + (x^3 - 36) = 0.$

$x > 30$, to make each group +ve.

∴ The lower limit of the real roots is – 30.

Hence the real roots lie between – 30 and 3. The rational roots of the equation (2) are the factors of 12 and hence they can be found among the values ±1, ±2, ±3, ±4, ±6, ±12.

From the limits of the real roots, it is enough to test for roots the divisors ±1, ±2, – 3, – 4, – 6, – 12

f (1) = 0, f(– 1) = 44.

Hence 1 is a root of the equation and – 1 is not a root of the equation. If 1 is added to the divisor we get 3, – 1, – 2, – 3, – 5, – 11.

3, – 3, – 5 are not the factor of 44.

Hence the divisors 2, – 4, – 6 are to be rejected. The remaining divisors are – 2, – 3, – 12. We shall apply Newton's method of divisors.

$$\begin{array}{r|rrrr} -2 & 12 & -23 & 10 & 1 \\ & & -6 & & \\ \hline & & -29 & & \end{array}$$

Therefore – 2 is not a root

$$\begin{array}{r|rrrr} -3 & 12 & -23 & 10 & 1 \\ & & -4 & 9 & \\ \hline & & -27 & 19 & \end{array}$$

Therefore – 3 is not a root

– 12	12	– 23	10	1
		– 1	2	– 1
		– 24	12	0

Therefore – 12 is a root of the equation .

Hence the rational roots of (2) are – 12 and 1. Since irrational roots and imaginary roots occur in pairs, the third roots is also rational. We can easily show that 1 is a repeated root of the equation (2). Hence the root of the original equation are $\frac{1}{2}, \frac{1}{2}, -6$.

Example 3. Solve the equation $3x^4 - 40x^3 + 130x^2 - 120x + 27 = 0$ given that it has two integral roots.

Solution.

The terms of the equation can be grouped as

$(3x^4 - 40x^3) + (130x^2 - 120x) + 27 = 0$

i.e., $x^3 (3x - 40) + 10x(13x - 12) + 27 = 0$.

Each group is positive if x = 14.

The upper limit of the real roots is 14.

Changing x into – x, we get the equation transformed into $3x^4 + 40x^3 + 130x^2 + 120x + 27 = 0$ whose lower limit of positive roots is zero. Hence the limits of the roots of the original equation are 0 and 14. The integral roots are found among the factors of 27, i.e., among the values ±1, ±3, ±9, ±27.

The real roots lies between 0 and 13.

Hence we to test for roots only the divisors 1, 3, 9.

f (1) = 0 , f (–1) = 320.

Hence 1 is a root of the equation and – 1 is not a root.

The divisors 3 and 9 increased by 1 are the factors of 320. Hence 3 and 9 are the roots. We shall apply Newton's method of divisors to find the quotient when the expression on the left side is dived by $(x - 1)$,$(x - 3)$,$(x - 9)$ and incidentally verify that 3,9 are the roots of equation.

1	27	–120	130	– 40	3
		27	– 93	37	– 3
		93	37	– 3	0

3	–27	93	– 37	3
		–9	28	–3
		84	– 9	0

9	9	– 28	3
		1	– 3
		27	0

Hence the quotient is $3x - 1$.

Hence the equation becomes $(x - 1)(x - 3)(x - 9)(3x - 1) = 0$

$\therefore$ The roots are 1, 3, 9 and $\frac{1}{3}$

Exercises

Solve the following equation, given that they commensurable roots

1. $x^3 - 5x^2 - 18x + 72 = 0$.
2. $x^4 - 39x^2 + 46x - 168 = 0$.
3. $x^5 - 12x^4 + 25x^3 - 48x^2 - 26x + 60 = 0$.
4. $2x^4 + x^3 - 2x^2 - 4x - 3 = 0$.

   Answers : 1.3,6, - 4, 2. 6, -7, $\frac{1\pm\sqrt{-15}}{3}$, 3. 1, - 1, 10, 4. $\frac{3}{2}, -1, \frac{-1\pm 3i}{3}$.

Horner's method

This method can be used to determine both the commensurable and the incommensurable roots of a numerical equation. First we shall explain the method for obtained the positive root. The procedure is to determine the root figure by figure, first the integral part and then the first decimal place, then the second decimal place and so on until the root terminates or the root has been obtained to the required degree of approximation. The main principle involved in this method is diminishing the roots by certain known quantities by successive transformations. In this method the successive transformations can be exhibited in a compact form and the roots can be obtained to any number of places of decimals required.

First we have to find by trial two consecutive integers between which a real positive root of the equation lies. This will give the integral part of the root. Let it be a first diminish all the roots of the equation by a. Then the transformed equation will have a root between 0 and 1. In order to avoid decimal in the working, all the roots of this transformed equation are multiplied by 10. Then the new transformed equation has a root between 0 and 10. By trial find the integers between which the root lies and thus find the integral part of the root. Let it be b. Then diminish the roots be b and again multiply the roots by 10 and continue the process till we get the root to the number of decimal we required.

Example 1. The equation $x^3 - 3x + 1 = 0$ has a root between 1 and 2. Calculate it to three places of decimals.

Solution.

Since the roots lies between 1 and 2, the integral part of the root is 1. Diminish the root of the equation by 1.

	1	0	– 3	1
(1				
		1	1	– 2
		1	– 2	- 1
		1	2	
		2	0	

$$\frac{1}{3}$$

The transformed equation is $x^3 + 3x^2 – 1 = 0$

This equation has therefore a root between 0 and 1.

Multiply the roots of this equation by 10.

Then the equation transforms into $x^3 + 30x^2 – 1000 = 0$

We can easily see that a root of this equation lies between 5 and 6. Diminish the roots of the equation by 5.

(5	1	30	0	– 1000
		5	175	875
		35	175	– 125
		5	200	
		40	375	
		5		
		45		

The transformed equation is $x^3 + 45x^2 + 375x – 125 = 0$.

This equation has therefore a root between 0 and 1.

Multiply the roots of the equation by 10.

Then the equation transforms into $x^3 + 450x^2 + 37500x – 125000 = 0$.

We can easily see that a root of this equation lies between 3 and 4.

Diminish the roots of this equation by 3.

1	450	37500	– 125000	(3

3	1359	116577
453	38859	– 8423
3	1368	
456	40227	
3		
459		

The transformed equation is $x^3 + 459x^2 + 40227x – 8423 = 0$.

Multiply the roots be 10.

Then the equation transforms into $x^3 + 4590x^2 + 4022700x – 8423000 = 0$.

We can easily see that a root of this equation lies between 2 and 3.diminish the root be 2

1	4590	4022700	– 8423000	(2
	2	9184	80637668	
	4592	4031884	– 359232	
	2	9188		
	4594	4041072		
	2			
	4596			

The transformed equation is $x^3 + 4596x^2 + 4041072x – 359232 = 0$

Multiply the roots by 10. Then the equation transforms into

$x^3 + 45960x^2 + 404107200x – 359232000 = 0$

We can easily see that a root of this equation lies between 0 and 1. We can stop with this since we require the root correct to three decimal places. Thus the root correct to three decimal places is 1.532. In the actual presentation we need write only the coefficients of the various transformed equations omitting completely the powers of x. The series of arithmetical operations is represented as follows:

1	0	– 3	1	(1.5320
	1	1	– 2	
	1	– 2	– 1000	
	1	2	875	
	2	0	– 125000	
	1	175	116577	
	30	175	– 8423000	
	5	200	8063768	
	35	37500	– 359232000	
	5	1359		
	40	38859		
	5	1368		
	450	4022700		
	3	9184		
	453	4031884		
	3	9188		
	456	401407200		
	3			
	4590			
	2			
	4592			

2

4594

2

45960

Example 2. Find the positive root of the equation $x^3 - 2x^3 - 3x - 4 = 0$ correct to three places of decimals.

Solution.

by Descartes' rule of signs, there can be at the most only one positive root and we can easily see that it lies between 3 and 4. The process is exhibited as follows:

-2	– 3	– 4	(3.2842
3	3	0	
1	0	– 4000	
3	12	2688	
4	1200	-1312000	
3	144	1242752	
70	1344	- 69248000	
2	148	64746224	
72	149200	– 4501776000	
2	6144	3243903688	
74	155344	– 1257872312	
2	6208		
760	16155200		
8	31356		

768	16186556
8	31392
776	1621794800
8	157044
7840	1621951844
4	
7844	
4	
7848	
4	
78520	
2	
78522	

∴ The roots correct to three decimal places is 3.284

Exercises

1. Find the positive root, correct to two decimal places of the equation $x^3 + 3x^2 + 2x - 5 = 0$.
2. Find the real root of $x^3 + 6x = 2$ to three places of decimals
3. Find the root between 0 and 1 correct to three places of decimal of the equation $x^3 + 18x - 6 = 0$.
4. Find the root of the equation $x^3 - 5x - 11 = 0$ which lies between 2 and 3 correct to two places of decimals.

   Answers : 1.0.90, 2.0.327, 3. 0.33, 4. 2.99.

UNIT III: SEQUENCE AND SERIES

Sequence and series : Sequence – limits, bounded, monotonic, convergent, oscillatory and divergent sequence – Algebra of limits – Subsequence – Cauchy sequence in R and Cauchy's general principle of convergence.

Sequences

Definition. Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a function and let $f(n) = a_n$. Then $a_1, a_2, a_3, \ldots, \ldots, a_n, \ldots$.. is called the sequences in $\mathbb{R}$ determined by the function f and is denoted by (a_n). a_n is called the n^{th} term of the sequence. The range of the function f which is a subset of $\mathbb{R}$, is called the range of the sequence

Examples.

1. The function $f : \mathbb{N} \rightarrow \mathbb{R}$ given by $f(n) = n$ determines the sequence 1, 2, 3, …, …, n, …
2. The function $f : \mathbb{N} \rightarrow \mathbb{R}$ given by $f(n) = n^2$ determines the sequence 1, 4, 9, …, …, n^2,…
3. The function $f : \mathbb{N} \rightarrow \mathbb{R}$ given by $f(n) = (-1)^n$ determines the sequence –1,1, –1, 1, …, Thus the terms of a sequence need not be distinct. The range of this sequence ie {1, –1}. Thus we see that the range of a sequence may be *finite* or *infinite*.
4. The sequence $((-1)^{n+1})$ is given by 1, –1, 1, –1… The range of this sequence is also {1, –1}. However we note that the sequence $((-1)^n)$ and $((-1)^{n+1})$ are different. The first sequence starts with –1 and the second sequence starts with 1.
5. The constant function $f : \mathbb{N} \rightarrow \mathbb{R}$ given by $f(n) = 1$ determines the sequence 1, 1, 1, …, … such a sequence is called a **constant sequence**
6. The function $f : \mathbb{N} \rightarrow \mathbb{R}$ given by

$$f(n) = \begin{cases} \frac{1}{2}\, n \text{ if } n \text{ is even} \\ \frac{1}{2}\,(1-n) \text{ if } n \text{ is odd} \end{cases}$$ determines the sequence $0, 1, -1, 2, -2, \ldots, n, -n,$ ….The range of this sequence is Z

7. The function f : $\mathbb{N} \to \mathbb{R}$ given by $f(n) = \frac{n}{n+1}$ determines the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$
8. The function $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = \frac{1}{n}$ determines the sequence $1\ \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots$
9. The function f : $\mathbb{N} \to \mathbb{R}$ given by $f(n) = 2n+3$ determines the sequence 5,7,9,11, …
10. Let $x \in \mathbb{R}$. The function f : $\mathbb{N} \to \mathbb{R}$ given by $f(n) = x^{n-1}$ determines the geometric sequence $1, x, x^2, \ldots\ldots x^n, \ldots\ldots$
11. The Sequence (– n) is given by –1, –2, –3,… – n,…… The range of this sequence is the set of all negative integers.
12. A sequence can also be described by specifying the first few terms and stating a rule for determining a_n in terms of the previous terms of the sequence. For example, let $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$. Then, $a_3 = a_2 + a_1 = 2$; $a_4 = a_3 + a_2 = 3$ and so on. We thus obtain the sequence 1,1,2, 3,5,8,………This sequence is called **Fibonacci's** sequence.

Bounded Sequences

Definition. A sequence (a_n) is said to be **bounded above** if there exists a real number k such that $a_n \le k$ for all **n** $\in \mathbb{N}$. k is called an upper bound of the sequence (a_n).

A sequence (a_n) is said to be **bounded below** if there exists a real number k such that $a_n \ge k$ for all n. k is called a **lower bound** of the sequence (a_n).

A sequence (a_n) is said to be a **bounded sequence** if it is both bounded above and below.

Note.

1. A sequence (a_n) is bounded if there exists a real number $k \ge 0$ such that $|a_n| \le k$ for all n

Examples.

1. Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots \frac{1}{n}, \ldots$ Here 1 is the l.u.b and 0 is the g. l.b. It is a bounded sequence.

2. The sequence 1,2,3, …….., n,…….. is bounded below but not bounded above. 1 is the g. l.b of the sequence.

3. The sequence –1, –2, –3,… – n, … is bounded above but not bounded below. –1

is the l.u.b of the sequence.

4. 1, –1, 1, –1, …. is a bounded sequence. 1 is the *l. u. b* –1 is the *g. l. b* of the sequence

5. Any constant sequence is a bounded sequence. Here *1.**u.b** **= g**. l. b* = the constant term of the sequence.

Monotonic sequence

Definition: A sequence (a_n) is said to be monotonic increasing if $a_n \leq a_n + 1$ for all *n*. (a_n) is said to be monotonic decreasing if $a_n \geq a_n + 1$ for all *n*. (a_n) is said to be strictly monotonic decreasing if $a_n < a_n + 1$ for all *n*. (a_n) is said to be monotonic if it is either monotonic increasing or monotonic decreasing.

Example.

1. 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, …. is a monotonic increasing sequence.
2. 1, 2, 3, 4 ….,n, ….. is a strictly monotonic increasing sequence.
3. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots$ is a strictly monotonic decreasing sequence.
4. The sequence (a_n) given by 1, –1, 1, –1, 1, … is neither monotonic increasing nor monotonic decreasing. Hence (a_n) is not a monotonic sequence.
5. $\left(\frac{2n-7}{3n+2}\right)$ is a monotonic increasing sequence.

Proof. $a_n - a_{n+1} = \frac{2n-7}{3n+2} - \frac{2(n+1)-7}{3(n+1)+2}$

$$= \frac{-25}{(3n+2)(3n+5)} < 0.$$

$\therefore a_n < a_n + 1.$

Hence the sequence is monotonic increasing.

6. Consider the sequence (a_n) where

$a_n = 1+\frac{1}{1!}+\frac{1}{2!}+ \ldots \frac{1}{n!}$ = Clearly (a_n) is a monotonic increasing sequence.

Note: A monotonic increasing sequence (a_n) is bounded below and q_1 is the g.l.b of the sequence. A monotonic decreasing sequence (a_n) is bounded above and a_1 is l. u. b of the sequence.

Solved Problem.

Show that if (a_n) is a monotonic sequence then $\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)$ is also a monotonic sequence.

Solution. Let (a_n) be a monotonic increasing sequence.

$$\therefore a_1 \leq a_2 \leq a_3 \ldots \leq a_n \leq \qquad \ldots\ldots (1)$$

Let $b = \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)$

Now, $b_{n+1} - b_n = \frac{a_1+a_2+\ldots+a_{n+1}}{n+1} - \frac{a_1+a_2+\ldots+a_n}{n}$

$$\geq \frac{na_{n+1}-(a_1+a_2+\ldots+a_n)}{n(n+1)}$$

$$= \frac{na_{n+1}-(a_n+a_n+\ldots+a_n)}{n(n+1)} \quad \text{by(1)}$$

$$= \frac{n(a_{n+1}-a_n)}{n(n+1)}$$

$$\geq 0.$$

$\therefore b_{n+1} \geq b_n$.

$\therefore$ (b_n) is monotonic increasing.

The proof is similar if (a_n) is monotonic decreasing.

Convergent sequences

Definition. A sequence (a_n) is said to converge to a number l if given $\epsilon > 0$ there exists a positive integer m such that $|a_n - l| < \epsilon$ for all $n \geq m$. We say that l is the limit of the sequence and we write $\lim_{n\to\infty} a_n = l$ or $(a_n) \to l$

Note.1 $(a_n) \to l$ iff given $\epsilon > 0$ there exists a natural number m such that $a_n \in (l - \epsilon, l + \epsilon,)$ for all $n \geq m$ i.e, All but a finite number of terms of the sequence lie within the interval $(l - \epsilon , l + \epsilon)$.

Note.2 The above definition does not give any method of finding the limit of a sequence. In many cases, by observing the sequence carefully. We can guess whether the limit exists or not and also the value of the limit.

Theorem 3.1. A sequence cannot converge to two different limits.

Proof. Let (a_n) be a convergent sequence.

If possible let l_1 and l_2 be two distinct limits of (a_n).

Let $\epsilon > 0$ be given.

Since $(a_n) \to l_1$, there exists a natural number n_1

such that $|a_n - l_1| < \frac{1}{2}\epsilon$ for all $n \geq n_1$ ....(1)

Since $(a_n) \to l_2$, there exists a natural number n_2

such that $|a_n - l_2| < \frac{1}{2}\epsilon$ for all $n \geq n_2$ --------- (2)

Let $m = \max \{n_1 , n_2\}$

Then $|l_1 - l_2| = |l_1 - a_m + a_m - l_2|$

$\leq |a_m - l_1| + |a_m - l_2|$

$< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon$ (by 1 and 2)

$= \epsilon$

$\therefore |l_1 - l_2| < \epsilon$ and this is true for every $\epsilon > 0$. Clearly this is possible only if $l_1 - l_2 = 0$. Hence $l_1 = l_2$

Examples

1. $\lim_{n \to \infty} \frac{1}{n} = 0$

   Proof. Let $\epsilon > 0$ be given. Then $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$ if $n > \frac{1}{\epsilon}$. Hence if we choose m to be any natural number such that $m > \frac{1}{\epsilon}$ then $\left|\frac{1}{n} - 0\right| < \epsilon$ for all $n \geq m$.

   $$\lim_{n \to \infty} \frac{1}{n} = 0$$

Note. If $\epsilon = 1/100$, then m can be chosen to be any natural number greater than 100. In this example the choice of m depends on the given ϵ and $[1/\epsilon] + 1$ is the smallest value of m that satisfies the requirements of the definition.

2. The constant sequence 1, 1, 1, …… converges to 1.

   Proof. Let $\epsilon > 0$ be given

   Let the given sequence be denoted by (a_n).

   Then $a_n = 1$ for all n.

   $\therefore |a_n - 1| = |1 - 1| = 0 < \epsilon$ for all $n \in N$.

   $\therefore |a_n - 1| < \epsilon$ for all $n \geq m$ where m can be chosen to be any natural number.

   $\therefore \lim_{n \to \infty} a_n = 1$

   Note. In this example, the choice of m does not depend on the given ϵ

3. $\lim_{n \to \infty} \frac{n+1}{n} = 1$

   Proof. Let $\epsilon > 0$ be given.

   Now, $\left|\frac{n+1}{n} - 1\right| = \left|1 + \frac{1}{n} - 1\right| = \left|\frac{1}{n}\right|$

∴ If we choose m to be any natural number greater than $\frac{1}{\epsilon}$ we have,

$$\left|\frac{n+1}{n} - 1\right| < \epsilon \text{ for all } n \geq m$$

$$\therefore \lim_{n \to \infty} \frac{n+1}{n} = 1$$

4. $\lim_{n \to \infty} \frac{1}{2^n} = 0$

Proof. Let $\epsilon > 0$ be given

Then $\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} < \frac{1}{n}$ ($\because 2^n > n \ \forall \ n \in N$).

∴ $\left|\frac{1}{2^n} - 0\right| < \epsilon$ for all $n \geq m$ where m is any natural number greater than $\frac{1}{\epsilon}$

$$\therefore \lim_{n \to \infty} \frac{1}{2^n} = 0$$

5. The sequence $((-1)^n)$ is not convergent

Proof. Suppose the sequence$((-1)^n)$ converges to l

Then, given $\epsilon > 0$, there exists a natural number m such that $|(-1)^n - l| < \epsilon$ for all $n > m$.

$$\therefore |(-1)^m - (-1)^{m+1}| = |(-1)^m - l + l - (-1)^{m+1}|$$

$$\leq |(-1)^m - l| + |(-1)^{m+1} - l|$$

$$< \epsilon + \epsilon = 2\epsilon$$

But $|(-1)^m - (-1)^{m+1}| = 2$.

∴$2 < 2\epsilon$ i.e., $1 < \epsilon$ which is a contradiction since $\epsilon > 0$ is arbitrary.

∴ The sequence $((-1)^n)$ is not convergent.

Theorem 3.2. Any convergent sequence is a bounded sequence.

Proof. Let (a_n) be a convergent sequence.

Let $\lim_{n \to \infty} a_n = l$.

Let $\epsilon > 0$ be given. Then there evists $m \in N$ such that $|a_n - l| < \epsilon$ for all $n \geq m$.

$\therefore |a_n| < |l| + \epsilon$ for all $n \geq m$.

Now, let $k = \max\{ |a_1|, |a_2| \ldots\ldots\ldots\ldots, |a_{m-1}|, |l| + \epsilon \}$

Then $|a_n| \leq k$ for all n.

$\therefore (a_n)$ is a bounded sequence.

Note. The converse of the above theorem is not true. For example, the sequence $((-1)^n)$ is a bounded sequence. However it is not a convergent sequence.

Divergent sequence

Definition. A sequence (a_n) is said to diverge to ∞ if given any real number $k > 0$, there exists $m \in N$ such that $a_n > k$ for all $n \geq m$. In symbols we write $(a_n) \to \infty$ or

$$\lim_{n \to \infty} a_n = \infty$$

Note. $(a_n) \to \infty$ if given any real number $k > 0$ there exists $m \in N$ such that $a_n \in (k, \infty)$ for all $n \geq m$

Examples

1. $(n) \to \infty$

Proof. Let $k > 0$ be any given real number.

Choose m to be any natural number such that $m > k$

Then $n > k$ for all $n \geq m$.

$\therefore (n) \to \infty$

2. $(n^2) \to \infty$

Proof. Let $k > 0$ be any given real number.

Choose m to be any natural number such that $m > \sqrt{k}$

Then $n^2 > k$ for all $n > m$

$\therefore (n^2) \to \infty$

3. $(2^n) \to \infty$

Proof. Let $k > 0$ be any given real number.

Then $2^n > k \Leftrightarrow n \log 2 > \log k$

$$\Leftrightarrow n > \frac{\log k}{\log 2}$$

Hence if we choose m to be any natural number such that $m > \frac{\log k}{\log 2}$,

then $2^n > k$ for all $n \geq m$

$\therefore (2^n) \to \infty$

Definition. A sequence (a_n) is said to diverge to $-\infty$ if given any real number $k < 0$ their exists $m \in N$ such that that $a_n < k$ for all $n \geq m$. In symbols we write

$$\lim_{n\to\infty} a_n = -\infty, \text{ or } (a_n) \to -\infty$$

Note. $(a_n) \to -\infty$ iff given any real number $k < 0$, there exists $m \in N$ such that $a_n \in (-\infty, k)$ for all $n \geq m$

A sequence (a_n) is said to be **divergent** if either $(a_n) \to \infty$ or $(a_n) \to -\infty$

Theorem 3.3. $(a_n) \to -\infty$ iff $(-a_n) \to -\infty$

Proof. Let $(a_n) \to \infty$

Let $k < 0$ be any given real number. Since $(a_n) \to \infty$ there exists $m \in N$ such that $a_n > -k$ for all $n \geq m$

$\therefore -a_n < k$ for all $n \geq m$

$\therefore$ $(-a_n) \to -\infty$.

Similarly we can prove that if $(-a_n) \to -\infty$ then $(a_n) \to \infty$.

Theorem 3.4. If $(a_n) \to \infty$ and $a_n \neq 0$ for all $n \in N$ then $(\frac{1}{a_n}) \to 0$.

Proof. Let $\epsilon > 0$ be given. Since $(a_n) \to \infty$, there exists $m \in N$ such that $a_n > \frac{1}{\epsilon}$ for all $n \geq m$

$\therefore$ $\frac{1}{a_n} < \epsilon$ for all $n \geq m$

$\therefore$ $\left|\frac{1}{a_n}\right| < \epsilon$ for all $n \geq m$.

$\therefore$ $\frac{1}{a_n} \to 0$.

Note. The converse of the above theorem is not true. For example, consider the sequence (a_n) where

$a_n = \frac{(-1)^n}{n}$ Clearly $(a_n) \to 0$

Now $(\frac{1}{a_n}) = (\frac{n}{(-1)^n}) = -1, 2, -3, 4, \ldots\ldots$ which neither converges nor diverges to ∞ or $-\infty$

thus if a sequence $(a_n) \to 0$, then the sequence $(\frac{1}{a_n})$ need not converge or diverge.

Theorem 3.5. If $(a_n) \to 0$ and $a_n > 0$ for all $n \in N$, then $(\frac{1}{a_n}) \to \infty$

Proof. Let $k > 0$ be any given real number.

Since $(a_n) \to 0$ there exists $m \in N$ such that $|a_n| < \frac{1}{k}$ for all $n \geq m$

$\therefore$ $a_n < \frac{1}{k}$ for all $n \geq m$ (since $a_n > 0$)

$\therefore$ $\frac{1}{a_n} > k$ for all $n \geq m$

$\therefore$ $(\frac{1}{a_n}) \to \infty$

Theorem 3.6. Any sequence (a_n) diverging to ∞ is bounded below but not bounded above.

Proof. Let $(a_n) \to \infty$. Then for any given real number $k > 0$ there exists $m \in \mathbb{N}$ such that $a_n > k$ for all $n \geq m$.
.........(1)

$\therefore$ k is not an upper bound of the sequence (a_n)

$\therefore$ (a_n) is not bounded above

Now let $l = \min\{a_1, a_2, \ldots a_m, k\}$.

From (1) we see that $a_n \geq l$ for all n.

$\therefore$ (a_n) is bounded below

Theorem 3.7. Any sequence (a_n) diverging to $-\infty$ is bounded above but not bounded below.

Proof is similar to that of theorem 3.6

Note 1. The converse of the above theorem is not true. For example, the function $f: \mathbb{N} \to \mathbb{R}$ defined by

$$f(n) = \begin{cases} 0 \text{ if n is odd} \\ \frac{1}{2} \text{n if n is even} \end{cases}$$

determines the sequence 0,1,0,2,0,3,..... which is bounded below and not bounded above. Also for any real number $k > 0$, we cannot find a natural number m such that $a_n > k$ for all $n \geq m$.

Hence this sequence does not diverge to ∞.

Similarly $f: \mathbb{N} \to \mathbb{R}$ *given by* $f(n) = \begin{cases} 0 \text{ } if \ n \ is \ odd \\ \frac{1}{2} \text{ n if n is even} \end{cases}$

Determines the sequence $0, -1, 0, -2, 0, \ldots\ldots$ which is bounded above and not bounded below. However this sequence does not diverge to $-\infty$.

Oscillating sequence

Definition . A sequence (a_n) which is neither convergent nor divergent to ∞ or $-\infty$ is said to be an oscillating sequence. An oscillating sequence which is bounded is said to be finitely oscillating. An oscillating sequence which is unbounded is said infinitely oscillating.

Examples.

1. Consider the sequence $((-1)^n)$. Since this sequence is bounded it cannot to ∞ or $-\infty$ (by theorems.6 and 7). Also this sequence is not convergent (by example 5 of theorem 4). Hence $((-1)^n)$ is a finitely oscillating sequence.

2. The function $f : \mathbb{N} \to \mathbb{R}$ defined by

$$f(n) = \begin{cases} \frac{1}{2} n \text{ if n is even} \\ \frac{1}{2}(1-n) \text{ if n is odd} \end{cases}$$

determines the sequence $0, 1, -1, 2, -2, 3, \ldots$.. The range of this sequence is **Z.** Hence it cannot converge or diverge to $\pm\infty$. This sequence is infinitely oscillating.

The Algebra of limits

In this section we prove a few simple theorems for sequences which are very useful in calculating limits of sequences.

Theorem 3.8. If $(a_n) \to$ a and $(b_n) \to$ b then $(a_n + b_n) \to$ a + b.

Proof. Let $\epsilon > 0$ be given.

Now $|a_n + b_n - a - b| = |a_n - a + b_n - b|$

$$\leq |a_n - a| + |b_n - b| \quad \ldots..(1)$$

Since $(a_n) \to$ a , there exist a natural number n_1 such that $|a_n - a| < \frac{1}{2}\epsilon$ for all $n \geq n_1$ ….(2)

Since $(b_n) \to$ b , there exist a natural number n_2 such that $|b_n - b| < \frac{1}{2}\epsilon$ for all $n \geq n_2$ ….(3)

Let $m = \max\{n_1, n_2\}$

Then $|a_n + b_n - a - b| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ for all $n \geq m$.

(by 1 , 2 and 3)

$\therefore$ $(a_n + b_n) \to$ a + b.

Note. Similarly we can prove that $(a_n - b_n) \to$ a − b.

Theorem 3.9. If $(a_n) \to$ a and k ∈ **R** then $(k\, a_n) \to$ k a.

Proof. If k = 0 , $(k\, a_n)$ is the constant sequence 0 , 0 , 0 , ... And hence the result is trivial.

Now , let k ≠ 0.

Then $|ka_n - ka| = |k|\, |a_n - a|$ ....(1)

Let $\epsilon > 0$ be given.

Since $(a_n) \to$ a , there exist m ∈ **N** such that

$|a_n - a| < \frac{\epsilon}{|k|}$ for all n ≥ m. ....(2)

$\therefore$ $|ka_n - ka| < \epsilon$ for all n ≥ m by (1 and 2).

$\therefore$ $(ka_n) \to$ ka.

Theorem 3.10. If $(a_n) \to$ a and $(b_n) \to$ b then $(a_n b_n) \to$ ab.

Proof. Let $\epsilon > 0$ be given.

Now , $|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab|$

$\leq |a_n b_n - a_n b| + |a_n b - ab|$

$= |a_n|\, |b_n - b| + |b|\, |a_n - a|$ ....(1)

Also, since $(a_n) \to$ a, (a_n) is a bounded sequences.

$\therefore$ There exist a real number k > 0 such that $|a_n| \leq$ k for all n. ....(2)

Using (1) and (2) we get

$|a_n b_n - ab| \leq$ k $|b_n - b| + |b|\, |a_n - a|$ .....(3)

Now since $(a_n) \to$ a , there exist a natural number n_1 such that

$|a_n - a| > \frac{\epsilon}{2|b|}$ for all n $\geq n_1$ ....(4)

Since $(b_n) \to$ b, there exist a natural number n_2 such that

$|b_n - b| < \frac{\epsilon}{2k}$ for all n $\geq n_2$ ....(5)

Let m = max$\{n_1, n_2\}$.

Then $|a_n b_n - ab| <$ k$\left(\frac{\epsilon}{2k}\right) + |b| \left(\frac{\epsilon}{2|b|}\right) = \epsilon$ for all n $\geq$ m (by 3 , 4 and 5)

Hence $(a_n b_n) \to$ ab.

Theorem 3.11. If $(a_n) \to$ a and $a_n \neq 0$ for all n and $a \neq 0$ then $\left(\frac{1}{a_n}\right) \to \frac{1}{a}$.

Proof. Let ϵ >0 be given

We have $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a_n - a}{a_n a}\right| = \frac{1}{|a_n|\,|a|} |a_n - a|$ ....(1)

Now , $a \neq 0$ Hence $|a| > 0$

Since $(a_n) \to$ a there exist $n_1 \in$ **N** such that

$|a_n - a| < \frac{1}{2}|a|$ for all n $\geq n_1$.

Hence $|a_n| > \frac{1}{2}|a|$ for all n $\geq n_1$. ....(2)

Using (1) and (2) we get

$\left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2}{|a|^2} |a_n - a|$ for all n $\geq n_1$ ....(3)

Now since $(a_n) \to$ a , there exist $n_2 \in$ **N** such that

$|a_n - a| < \frac{1}{2}\epsilon\, |a|^2$for all n $\geq n_2$. ....(4)

Let m = max$\{n_1, n_2\}$.

$\therefore \left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2}{|a|^2} \frac{|a|^2 \epsilon}{2} = \epsilon$ for all n $\geq$ m (by 3 and 4)

$\therefore \left(\frac{1}{a_n}\right) \to \frac{1}{a}$.

Corollary. Let $(a_n) \to$ a and $(b_n) \to$ b where $b_n \neq 0$ for all n and b $\neq$ 0.

Then $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$.

Proof. $\left(\frac{1}{b_n}\right) \to \frac{1}{b}$. (by theorem 3.11)

$\therefore$ $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$. (by theorem 3.10)

Note. Even if $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ do not exist, $\lim_{n\to\infty}(a_n + b_n)$, $\lim_{n\to\infty}(a_n b_n)$ and $\lim_{n\to\infty}(\frac{a_n}{b_n})$ may exist. For example let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Clearly $\lim_{n\to\infty} a_n$and $\lim_{n\to\infty} b_n$ do not exist. Now $(a_n + b_n)$ is the constant sequence 0 , 0 , 0 , ….. Each of $(a_n b_n)$ and (a_n/b_n) is the constant sequence -1 , -1 , ….. Hence $(a_n + b_n) \to 0$. $(a_n b_n) \to -1$ and $(a_n/b_n) \to -1$.

Theorem 3.12. If $(a_n) \to$ a then $(|a_n|) \to |a|$.

Proof. Let $\epsilon > 0$ be given

Now $\big||a_n| - |a|\big| \leq |a_n - a|$ …………(1)

Since $(a_n) \to$ a there exist m$\in$ $\boldsymbol{N}$ such that $|a_n - a| < \epsilon$ for all n $\geq$ m.

Hence from (1) we get $\big||a_n| - |a|\big| < \epsilon$ for all n $\geq$ m.

Hence $(|a_n|) \to |a|$.

Theorem 3.13. If $(a_n) \to$ a and $a_n \geq 0$ for all n then a $\geq$ 0.

Proof. Suppose a $<$ 0 . Then $-$a $>$ 0.

Choose ϵ such that 0 $< \epsilon <$ $-$a so that a + ϵ $<$ 0.

Now , since $(a_n) \to$ a , there exist m $\in$ **N** such that $|a_n - a| < \epsilon$ for all n $\leq$ m.

$\therefore$ a$-\epsilon < a_n <$ a+ϵ for all n $\leq$ m.

Now, since a+ϵ $<$ 0 , we have $a_n < 0$ for all n $\geq$ m which is a contradiction since $a_n \geq 0$.

$\therefore$ a $\geq$ 0.

Note. In the above theorem if $a_n > 0$ for all n , we cannot say that a > 0 . For example consider the sequence $\left(\frac{1}{n}\right)$. Here $\frac{1}{n} > 0$ for all n and $\left(\frac{1}{n}\right) \to 0$.

Theorem 3.14. If $(a_n) \to$ a , $(b_n) \to$ b and $a_n \leq b_n$ for all n , then a ≤ b.

Proof. Since $a_n \leq b_n$, we have $b_n - a_n \geq 0$ for all n.

Also ($b_n - a_n$)→ b – a (by theorem 3.8).

∴ b – a ≥ 0 (by theorem 3.13)

∴ b ≥ a.

Theorem 3.15. If $(a_n) \to l$, $(b_n) \to l$ and $a_n \leq c_n \leq b_n$ for all n, then $(c_n) \to l$.

Proof. Let $\epsilon > 0$ be given.

Since $(a_n) \to l$, there exist $n_1 \in$ **N** such that $l - \epsilon < a_n < l + \epsilon$ for all n ≥ n_1.

Similarly, there exist $n_2 \in$ **N** such that $l - \epsilon < b_n < l + \epsilon$ for all n ≥ n_2.

Let m = max$\{n_1, n_2\}$.

∴ $l - \epsilon < a_n \leq c_n \leq b_n < l + \epsilon$ for all n ≥ m.

∴ $l - \epsilon < c_n < l + \epsilon$ for all n ≥ m.

∴ $|c_n - l| < \epsilon$ for all n ≥ m.

∴ $(c_n) \to l$.

Theorem 3.16. If $(a_n) \to a$ and $a_n \geq 0$ for all n and $a \neq 0$, then $\left(\sqrt{a_n}\right) \to \sqrt{a}$.

Proof. Since $a_n \geq 0$ for all n, a ≥ 0.(by theorem 3.13)

Now, $\left|\sqrt{a_n} - \sqrt{a}\right| = \left|\frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}\right|$.

Since $(a_n) \to a \neq 0$. as in theorem 11 we obtain $a_n > \frac{1}{2}$ a for all n ≥ n_1

∴ $\sqrt{a_n} > \sqrt{\left(\frac{1}{2}\text{ a }\right)}$ for all n ≥ n_1.

$\therefore \left|\sqrt{a_n} - \sqrt{a}\right| < \frac{\sqrt{2}}{(\sqrt{2}+1)\sqrt{a}} |a_n - a|$ for all n $\geq n_1$ ....(1)

Now, let $\epsilon > 0$ be given.

Since $(a_n) \rightarrow a$, there exist $n_2 \in \mathbf{N}$ such that

$|a_n - a| < \epsilon \sqrt{a}\ (\sqrt{2} + 1)/\sqrt{2}$ for all n $\geq n_2$ ....(2)

Let m = max$\{n_1, n_2\}$.

Then $\left|\sqrt{a_n} - \sqrt{a}\right| < \epsilon$ for all n ≥ m (by 1 and 2).

$\therefore \left(\sqrt{a_n}\right) \rightarrow \sqrt{a}$.

Theorem 3.17. If $(a_n) \rightarrow \infty$ and $(b_n) \rightarrow \infty$ then $(a_n + b_n) \rightarrow \infty$.

Proof. Let k > 0 be any given real number.

Since $(a_n) \rightarrow \infty$, there exists $n_1 \in \mathbf{N}$ such that $a_n > \frac{1}{2}$k for all n $\geq n_1$.

Similarly there exists $n_2 \in \mathbf{N}$ such that $b_n > \frac{1}{2}$k for all n $\geq n_2$.

Let m = max$\{n_1, n_2\}$.

Then $a_n + b_n >$ k for all n ≥ m.

$\therefore (a_n + b_n) \rightarrow \infty$.

Theorem 3.18. If $(a_n) \rightarrow \infty$ and $(b_n) \rightarrow \infty$ then $(a_n b_n) \rightarrow \infty$.

Proof. Let k > 0 be any given real number.

Since $(a_n) \rightarrow \infty$, there exist $n_1 \in \mathbf{N}$ such that $a_n > \sqrt{k}$ for all n $\geq n_1$.

Similarly there exists $n_2 \in \mathbf{N}$ such that $b_n > \sqrt{k}$ for all n $\geq n_2$.

Let m = max$\{n_1, n_2\}$.

Then $a_n b_n >$ k for all n ≥ m.

$\therefore (a_n b_n) \rightarrow \infty$.

Theorem 3.19. Let $(a_n) \to \infty$ then

(i) If c >0 , $(c\, a_n) \to \infty$

(ii) If c < 0 , $(c\, a_n) \to -\infty$

Proof. (i) Let c > 0.

Let k > 0 be any given real number.

Since $(a_n) \to \infty$, there exist m ∈ ***N*** such that $a_n > \frac{k}{c}$ for all n ≥ m.

∴ $c\, a_n > k$ for all n ≥ m.

∴ $(c\, a_n) \to \infty$.

(ii) Let c < 0 . Let k < 0 be any given real number. Then $\frac{k}{c} > 0$.

∴ There exists m ∈ **N** such that $a_n > \frac{k}{c}$ for all n ≥ m.

∴ $c\, a_n < k$ for all n ≥ m (since c < 0).

∴ $(c\, a_n) \to -\infty$.

Theorem 3.20. If $(a_n) \to \infty$ and (b_n) is bounded then $(a_n + b_n) \to \infty$.

Proof.

Since (b_n) is bounded, there exists a real number m < 0 such that $b_n > m$ for all n. .....(1)

Now, let k > 0 be any real number.

Since m < 0 , k – m > 0.

Since $(a_n) \to \infty$, there exists $n_0 \in$ **N** such that $a_n > k - m$ for all $n \geq n_0$. ....(2)

∴ $a_n + b_n > k - m + m = k$ for all $n \geq n_0$ (by 1 and 2).

∴ $(a_n + b_n) \to \infty$.

Solved Problems.

1. Show that $\lim_{n\to\infty} \frac{3n^2+2n+5}{6n^2+4n+7} = \frac{1}{2}$

Solution. $a_n = \frac{3n^2+2n+5}{6n^2+4n+7} = \frac{3+\frac{2}{n}+\frac{5}{n^2}}{6+\frac{4}{n}+\frac{7}{n^2}}$

Now , $\lim_{n\to\infty}\left(3+\frac{2}{n}+\frac{5}{n^2}\right)$

$= 3 + 2\lim_{n\to\infty}\frac{1}{n} + 5\lim_{n\to\infty}\frac{1}{n^2}.$

$= 3 + 0 + 0 = 3$

Similarly , $\lim_{n\to\infty}\left(6+\frac{4}{n}+\frac{7}{n^2}\right) = 6$

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3+\frac{2}{n}+\frac{5}{n^2}}{6+\frac{4}{n}+\frac{7}{n^2}}$

$= \frac{\lim_{n\to\infty} 3+\frac{2}{n}+\frac{5}{n^2}}{\lim_{n\to\infty} 6+\frac{4}{n}+\frac{7}{n^2}}$

$= \frac{3}{6} = \frac{1}{2}.$

2. Show that $\lim_{n\to\infty}\left(\frac{1^2+2^2 \ldots + n^2}{n^2}\right) = \frac{1}{3}$.

Solution. We know that $1^2 + 2^2 \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

$\therefore \lim_{n\to\infty}\frac{1^2+2^2 \ldots + n^2}{n^2} = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^2}.$

$= \lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$= \frac{1}{3}.$

3. Show that $\lim_{n\to\infty}\frac{n}{\sqrt{(n^2+1)}} = 1.$

Solution. $\lim_{n\to\infty}\frac{n}{\sqrt{(n^2+1)}} = \lim_{n\to\infty}\frac{1}{\sqrt{\left(1+\frac{1}{n^2}\right)}}$

$= \frac{1}{\lim_{n\to\infty}\sqrt{\left(1+\frac{1}{n^2}\right)}}$ (By theorem 3.11)

$$= \frac{1}{\sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)}} \quad \text{(By theorem 3.16)}$$

$$= 1$$

4. Show that if $(a_n) \to 0$ and (b_n) is bounded, then $(a_n b_n) \to 0$.

Solution. Since (b_n) is bounded, there exists k > 0 such that $|b_n| \leq$ k for all n.

$\therefore$ $|a_n b_n| \leq$ k $|a_n|$.

Now, let $\epsilon > 0$ be given.

Since $(a_n) \to 0$ there exists m ∈ **N** such that $|a_n| < \frac{\epsilon}{k}$ for all n ≥ m.

$\therefore$ $|a_n b_n| < \epsilon$ for all n ≥ m.

$\therefore$ $(a_n b_n) \to 0$.

5. Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

Solution. $|\sin n| \leq 1$ for all n.

$\therefore$ $(\sin n)$ is a bounded sequences

Also , $\left(\frac{1}{n}\right) \to 0$.

$\therefore$ $\left(\frac{\sin n}{n}\right) \to 0$ (by problem 3.4).

6. Show that $\lim_{n\to\infty} (a^{1/n}) = 1$ where a > 0 is any real number.

Solution. Case (i) Let a = 1 . Then $a^{1/n}$ =1 for each n . Hence $(a^{1/n}) \to 1$

Case (ii) Let a > 1. Then $a^{1/n}$ >1.

Let $a^{1/n} = 1 + h_n$ where $h_n > 0$.

$\therefore$ $a = (1 + h_n)^n$

$= 1 + nh_n + \ldots\ldots + {h_n}^n$.

$>1 + nh_n$.

$\therefore h_n < \frac{a-1}{n}$.

$\therefore\ 0 < h_n < \frac{a-1}{n}$.

Hence $\lim\limits_{n\to\infty} h_n = 0$.

$\therefore\ (a^{1/n}) = (1 + h_n) \to 1$.

Case (iii) Let 0 < a < 1. Then $\frac{1}{a} > 1$.

$\therefore \left(\frac{1}{a}\right)^{1/n} \to 1$ (By case (ii))

$\therefore\ \left(\frac{1}{a^{1/n}}\right) \to 1$.

$\therefore\ (a^{1/n}) \to 1$.

7. Show that $\lim_{n\to\infty} (n)^{\frac{1}{n}} = 1$.

Solution. Clearly $n^{1/n} \geq 1$ for all n.

Let $n^{1/n} = 1 + h_n$ where $h_n \geq 0$.

Then n = $(1 + h_n)^n$

$= 1 + nh_n + nc_2\ h_n^2 + \ldots.. + h_n^n$.

$> \frac{1}{2}$ n(n – 1) h_n^2

$\therefore\ h_n^2 < \frac{2}{(n-1)}$

$\therefore\ h_n < \sqrt{\frac{2}{n-1}}$.

Since $\sqrt{\frac{2}{n-1}} \to 0$ and $h_n \geq 0$, $(h_n) \to 0$.

$\therefore\ (n^{1/n}) = (1 + h_n) \to 1$.

8. Show that $\lim_{n\to\infty}\left(\frac{1}{\sqrt{(2n^2+1)}}+\frac{1}{\sqrt{(2n^2+2)}}+\cdots+\frac{1}{\sqrt{(2n^2+n)}}\right)=\frac{1}{\sqrt{2}}$.

Solution.

Let $a_n=\frac{1}{\sqrt{(2n^2+1)}}+\frac{1}{\sqrt{(2n^2+2)}}+\ldots\ldots+\frac{1}{\sqrt{(2n^2+n)}}$.

Then we have the inequality

$$\frac{n}{\sqrt{(2n^2+n)}}<a_n<\frac{n}{\sqrt{(2n^2+1)}}.$$

$$\therefore\ \frac{1}{\sqrt{\left(2+\frac{1}{n}\right)}}<a_n<\frac{1}{\sqrt{\left(2+\frac{1}{n^2}\right)}}.$$

Now, $\lim_{n\to\infty}\frac{1}{\sqrt{\left(2+\frac{1}{n}\right)}}=\lim_{n\to\infty}\frac{1}{\sqrt{\left(2+\frac{1}{n^2}\right)}}=\frac{1}{\sqrt{2}}$.

$\therefore\ \lim_{n\to\infty} a_n=\frac{1}{\sqrt{2}}$. (By theorem 3.15).

9. Give an example to show that if (a_n) is a sequence diverging to ∞ and (b_n) is a sequence diverging to $-\infty$ then (a_n+b_n) need not be a divergent sequence.

Solution. Let $(a_n)=(n)$ and $(b_n)=(-n)$.

Clearly $(a_n)\to 0$ and $(b_n)\to-\infty$.

However (a_n+b_n) is the constant sequence 0 , 0 , 0 ,…. Which converges to 0.

Exercises.

1.Evaluate the limits of the following sequences as n → ∞.

(a) $\left(\frac{(n^2+3)(n^3+9)}{(n+1)\,(n^4+6)}\right)$

(b) $\frac{\sqrt{(3n^2-5n+4)}}{2n-7}$

(c) $\left(\frac{1+2+3+\cdots+n)}{n^2}\right)$

(d) $\left(\frac{1^3+2^3+\cdots+n^3)}{n^4}\right)$

2.A sequence (a_n) is called a **null sequence** if $(a_n) \to 0$. Show that if (a_n) and (b_n) are null sequences then $(a_n + b_n)$, $(a_n b_n)$, $(\mathrm{k}a_n)$ and $(|a_n|)$are also null sequences.

3.If $(a_n) \to -\infty$ and $(b_n) \to -\infty$, then show that $(a_n + b_n) \to -\infty$ and $(a_n b_n) \to \infty$.

4.Prove the following.

(a) $\lim_{n\to\infty} \left(\frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \cdots + \frac{1}{\sqrt{(n^2+n)}}\right) = 1.$

5.Give examples of sequences (a_n) and (b_n) such that

(a) $(a_n) \to \infty$ $(b_n) \to \infty$ and $(a_n - b_n)$ converges to 5.

(b) $(a_n) \to \infty$ $(b_n) \to \infty$ and $(a_n - b_n) \to \infty$.

(c) $(a_n) \to l$, $(b_n) \to \infty$ and $(a_n b_n) \to -\infty$.

Answers : 1.(a). $\frac{1}{2}$,(b). $\sqrt{\frac{3}{2}}$,(c). $\frac{1}{2}$,(d). $\frac{1}{4}$,

Subsequences

Definition. Let (a_n) be a sequence. Let (n_k) be a strictly increasing sequence of natural numbers. Then (a_{nk}) is called a subsequence of (a_n).

Note. The terms of a subsequences occur in the same order in which they occur in the original sequence.

Examples.

1. (a_n) is a subsequence of any sequence (a_n). Note that in this example the interval between any two terms of the subsequence is the same, (i.e.,) n_1=2 , n_2=4 , n_3=6 ,... n_k = 2k.

2. (a_{n^2}) is a subsequence of any sequence (a_n). Hence $a_{n1} = a_1$, $a_{n2} = a_4$, $a_{n3} = a_9$ ….. Here the interval between two successive terms of the subsequence goes on increasing as k becomes large. Thus the interval between various terms of a subsequence need not be regular.

3. Any sequence (a_n) is a subsequence of itself.

4. Consider the sequence(a_n) given by 1, 0, 1, 0 …. Now, (a_n) given by 1, 1, 1, ….. is a sequence of (a_n) . Here (a_n) is not convergent whereas the subsequence (b_n) converges to 1. Thus a subsequence of a non-convergent sequence can be a convergent sequence.

Note. A subsequences of a given subsequence (a_{nk}) of a sequence (a_n) is again a subsequence of (a_n).

Theorem 3.21. If a sequence (a_n) converges to l . then every subsequence(a_{nk}) of (a_n) also converges to l.

Proof. Let $\epsilon > 0$ be given.

Since $(a_n) \to l$ there exists m ∈ $\boldsymbol{N}$ such that

$|a_n - l| < \epsilon$ for all n ≥ m. …….(1)

Now choose $n_{k0} \geq$ m.

Then k ≥ $k_0 \Rightarrow n_k \geq n_{k0}$ ($\because (n_k)$ is monotonic increasing)

$\Rightarrow n_k \geq$ m.

$\Rightarrow \;|a_{nk} - l| < \epsilon$ (by 1)

Thus $|a_{nk} - l| < \epsilon$ for all k ≥ k_0.

$\therefore (a_{nk}) \to l$.

Note 1. If a subsequence of a sequence converges, then the original sequence need not converges, then the original sequence need not converge.

Note 2. If a sequence (a_n) has two subsequences converging to two different limits, then (a_n) does not converge. For example, consider the sequence (a_n) given by

$$a_n \begin{cases} \frac{1}{n} \text{ if n is even} \\ 1 + \frac{1}{n} \text{ if n is odd} \end{cases}$$

Here the subsequence $(a_{2n}) \to 0$ and the subsequence $(a_{2n}) \to l$. Hence the given sequence (a_{2n})does not converge.

Theorem 3.22. If the subsequences (a_{2n-1}) and (a_{2n}) of a sequence (a_n) converge to the same limit l then (a_n) also converges to l .

Proof. Let $\epsilon > 0$ be given. Since $(a_{2n-1}) \to l$ there exists $n_1 \in \boldsymbol{N}$ such that $|a_{2n-1} - l| < \epsilon$ for all $2n - 1 \geq n_1$.

Similarly there exists $n_2 \in \boldsymbol{N}$ such that $|a_{2n} - l| < \epsilon$ for all $2n \geq n_2$.

Let $\mathrm{m} = \max\{n_1, n_2\}$.

Clearly $|a_n - l| < \epsilon$ for all $\mathrm{n} \geq \mathrm{m}$.

$\therefore (a_n) \to l$.

Note. The above result is true even if we have $l \to \infty$ $or - \infty$.

Definition. Let (a_n) be a sequence. A natural number m is called a **peak point** of the sequence (a_n) if $a_n < a_n$ for all $\mathrm{n} > \mathrm{m}$.

Example.

1. For the sequence (½), every natural number is a peak point and hence the sequence has infinite number of peak point. In general for a strictly monotonic decreasing sequence every natural number is a peak point.
2. Consider the sequence $1 , \frac{1}{2} , \frac{1}{3} , -1 , -1 , \ldots\ldots$ Here 1 , 2, 3 are the peak points of the sequence.
3. The sequence 1 , 2 , 3 , …… has no peak point. In general a monotonic increasing sequence has no peak point.

Theorem 3.23. Every sequence (a_n) has no monotonic subsequence.

Proof. Case (i) (a_n) has infinite number of peak points. Let the peak points be

$n_1 < n_2 < \ldots . < n_k < \ldots\ldots$ Then $a_{n1} > a_{n2} > \ldots .. > a_{nk} > \ldots .$

$\therefore (a_{nk})$ is a monotonic decreasing subsequence of (a_n).

Case (ii) (a_n) has only a finite number of peak points or no peak points.

Choose a natural number n_1 such that there is no peak point greater than or equal to n_1. Since n_1 is not a peak point of (a_n) , there exists $n_2 > n_1$ such that $a_{n1} \geq a_{n2}$. Again since n_2 is not a peak point , there exist : $n_3 > n_2$ such that $a_{n3} \geq a_{n2}$.Repeating this process we get a monotonic increasing subsequence (a_{nk}) of (a_n).

Theorem 3.24. Every bounded sequences has a convergent subsequences.

Proof. Let (a_n) be a bounded sequence. Let (a_{nk}) be monotonic subsequence of (a_n) since (a_n) is bounded (a_{nk}) is also bounded.

$\therefore (a_{nk})$ is a bounded monotonic sequence and hence converges.

$\therefore (a_{nk})$ is a convergent subsequence of (a_n).

Exercises.

1. Prove that if a sequence (a_n) diverges to ∞ then every subsequence of (a_n) also diverges to ∞.
2. Prove that if a sequence (a_n) diverges to $-\infty$ then every subsequence of (a_n) also converges to $-\infty$.
3. Give examples of (i) a sequence which does not diverge to but ∞ ha s a subsequence diverging to ∞ (ii) a sequence which does not diverge to $-\infty$ but has a subsequence diverging to $-\infty$ (iii) a sequence (a_n) having two subsequences, one converging to ∞ and the other diverging $-\infty$.
4. Prove that each of the following sequences is not convergent by exhibiting two subsequences converging to two different limits.
   - (i) $1 , \frac{1}{2}, 1 , \frac{1}{3}, 1 , \frac{1}{4}, \ldots\ldots 1 , \frac{1}{n}, \ldots\ldots$
   - (ii) $1 , 2, 1 , 3 , 1 , 4 , \ldots\ldots$
   - (iii) $((-1^n))$

Cauchy sequences.

Definition. A sequence (a_n) is said to be a **Cauchy sequence** if given $\epsilon > 0$, there exists $n_0 \in \boldsymbol{N}$ such that $|a_n - a_m| < \epsilon$ for all n , m $\geq n_0$.

Note. In the above definition the condition $|a_n - a_m| < \epsilon$ for all n , m $\geq n_0$ can be written in the following equivalent form, namely , $|a_{n+p} - a_n| < \epsilon$ for all n $\geq n_0$ and for all positive integers p.

Examples

1. The sequence $\left(\frac{1}{n}\right)$ is a Cauchy sequence

Proof. Let $(a_n) = \left(\frac{1}{n}\right)$. Let $\epsilon > 0$ be given. Now, $|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right|$.

$\therefore$ If we choose n_0 to be any positive integer greater than $1/\epsilon$, we get

$|a_n - a_m| < \epsilon$ for all n , m $\geq n_0$.

$\therefore$ $\left(\frac{1}{n}\right)$ is a Cauchy sequence.

2. The sequence $((-1^n))$ is not a Cauchy sequence.

Proof. Let $(a_n) = ((-1^n))$.

$\therefore$ $|a_n - a_{n+1}| = 2$.

$\therefore$ If $\boldsymbol{\epsilon} < 2$, we cannot find n_0 such that $|a_n - a_{n+1}| < \epsilon$ for all n $\geq n_0$.

$\therefore$ $((-1^n))$ is not a Cauchy sequence.

3. (n) is not a Cauchy sequence.

Proof. Let $(a_n) = $ (n).

$\therefore$ $|a_n - a_m| \geq 1$ if n $\neq$ m.

$\therefore$ If we choose $\boldsymbol{\epsilon} < 1$, we cannot find n_0 such that $|a_n - a_m| < \epsilon$ for all n , m $\geq n_0$.

$\therefore$ (n) is not a Cauchy sequence.

Theorem 3.25. Any convergent sequence is a Cauchy sequence.

Proof.

Let $(a_n) \to l$. Then given $\boldsymbol{\epsilon} > 0$, there exists $n_0 \in \boldsymbol{N}$ such that $|a_n - l| < \frac{1}{2}\epsilon$ for all n $\geq n_0$.

$\therefore\ |a_n - a_m| = |a_n - l + l - a_m|$

$< |a_n - l| + |l - a_m|$

$< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ for all n , m $\geq n_0$.

$\therefore\ (a_n)$ is Cauchy sequence.

Theorem 3.26. Any Cauchy sequence is a bounded sequence is bounded sequence.

Proof. Let (a_n) be a Cauchy sequence.

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \boldsymbol{N}$ such that $|a_n - a_m| < \epsilon$ for all n , m $\geq n_0$.

$\therefore\ |a_n| < |a_{n0}| + \epsilon$ for n $\geq n_0$.

Now , let k = max { $|a_1|$, $|a_2|$, …. $|a_{n0}|+ \epsilon$ } .

Then $|a_n| \leq$ k for all n.

$\therefore\ (a_n)$ is a bounded sequence.

Theorem 3.27. Let (a_n) be a Cauchy sequence. If (a_n) has a sequence (a_{nk}) converging to , then $(a_n) \to l$.

Proof.

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \boldsymbol{N}$ such that $|a_n - a_m| < \epsilon$ for all n , m $\geq n_0$ ….(1)

Also since $(a_{nk}) \to l$, there exists $k_0 \in \boldsymbol{N}$ such that

$|a_{nk} - l| < \frac{1}{2}\epsilon$ for all k $\geq k_0$ …………..(2)

Choose n_k such that $n_k > n_{k0}$ and n_0

Then $|a_n - l| = |a_n - a_{nk} + a_{nk} - l|$

$\leq |a_n - a_{nk}| + |a_{nk} - l|$

$< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ for all n $\geq n_0$.

Hence $(a_n) \to l$.

Theorem 3.28 (Cauchy's general principle of convergence)

A sequence (a_n) in **R** is convergent iff it is a Cauchy sequence.

Proof. In theorem 25 we have proved that any convergent sequence is a Cauchy sequence.

Conversely, let (a_n) be a Cauchy sequence in **R.**

$\therefore$ (a_n) is a bounded sequence (by theorem 26)

$\therefore$ There exist a subsequence (a_{nk}) of (a_n) such that $(a_{nk}) \to l$ (by theorem 23)

$\therefore$ $(a_n) \to l$ (by theorem 27).

Revision questions on chapter 3.

Determine which of the following statements are true and which are false.

1. The range of a sequence is an infinite set.
2. Two sequences are equal if they have the same range.
3. Any convergent sequence is bounded.
4. Any bounded sequence is convergent.
5. Any monotonic sequence is bounded.
6. Any monotonic sequence is convergent.
7. Any bounded monotonic sequence is convergent.
8. Any monotonic sequence which is not bounded is divergent.
9. Any monotonic sequence cannot oscillate.
10. Sum of two convergent sequences is again a convergent sequence.
11. Sum of two divergent sequences is again a divergent sequence.
12. Sum of two monotonic sequences is a monotonic sequence.
13. Sum of two monotonic increasing sequences is a monotonic increasing sequence.
14. Sum of two oscillating sequences is again an oscillating sequence.
15. A constant sequence is both monotonic increasing and monotonic decreasing.
16. An oscillating sequence is always bounded.
17. Any constant sequence is convergent.

18. If $(a_n)\to 0$ then $(1/a_n) \to \infty$.

19. If $(a_n)\to 0$ and $a_n > 0$ for all n, then $(1/a_n)$ diverges to ∞.

20. If $(a_n)\to 0$ and $a_n < 0$ for all n, then $(1/a_n)$ diverges to $-\infty$.

21. If $(a_n)\to \infty$ and $a_n \neq 0$ for all n, then $(1/a_n) \to 0$.

22. If $(a_n)\to \infty$ and $(ca_n) \to \infty$.

23. If $(a_n)\to \infty$ and c > 0, then $(ca_n) \to \infty$.

24. Any convergent sequence is a Cauchy sequence.

25. Any Cauchy sequence of real numbers is convergent.

26. Any Cauchy sequence is bounded.

27. Every sequence has infinitely many subsequences.

28. Any subsequence of a convergent sequence is convergent.

29. Every sequence has a convergent subsequence.

30. Every bounded sequence has a converge subsequence.

31. Every sequence has a monotonic subsequence.

32. Every sequence has a limit point.

33. Every sequence has a finite limit point.

34. Every bounded sequence has a finite limit point.

35. Every sequence has a finite number of limit point of the sequence.

36. The limit of a convergent sequence is a limit point of the sequence.

37. If a is a limit point of a sequence (a_n) , then $(a_n)\to a$.

38. If a is a only limit point of a sequence (a_n) then $(a_n)\to a$.

39. If a is a limit point of a sequence (a_n), then there exists a subsequence converging to a.

40. Every sequence has an upper limit.

41. Every sequence has a lower limit.

42. For any sequence (a_n) , lower limit $a_n <$ upper limit (a_n).

43. A sequence $(a_n) \to a$ iff lower limit $(a_n) =$ upper limit $(a_n) = a$.

44. $\overline{\lim}\,(a_n + b_n) = \overline{\lim}\,a_n + \overline{\lim}\,b_n$.

Answers.

1, 2, 4, 5, 6, 11, 12, 14, 16, 18, 22, 29, 33, 35, 37, 44, are false 3, 7 to 10, 13, 15, 17, 19, 20, 21, 23, to 28, 30 to 32, 34, 36, 38, to 43 are true.

UNIT IV: SERIES

Series – convergence, divergence – geometric, harmonic, exponential, binomial and logarithmic series – Cauchy's general principle of convergence – Comparison test – tests of convergence of positive termed series – Kummer's test, ratio test, Raabe's test, Cauchy's root test, Cauchy's condensation test.

Infinite series

Definition. Let $(a_n) = a_1, a_2, \ldots\ldots a_n, \ldots\ldots$ be a sequence of real numbers. Then the formal expression $a_1 + a_2 + \ldots\ldots + a_n + \ldots\ldots$ is called an infinite series of real numbers and is denoted by $\sum_1^\infty a_n$ or $\sum a_n$.

Let $s_1 = a_1$; $s_2 = a_1 + a_2$; $s_3 = a_1 + a_2 + a_3 + \ldots.$ $s_n = a_1 + a_2 + \cdots + a_n$.

Then (s_n) is called the sequence of partial sums of the given series $\sum a_n$.

The series $\sum a_n$ is said to converge, diverge or oscillate according as the sequence of partial sums (s_n) converges, diverges or oscillates.

If $(s_n) \to s$, we say that the series $\sum a_n$ converges to the sum s.

We note that the behavior of a series does not change if a finite number of terms are added or altered.

Examples.

1. Consider the series $1 + 1 + 1 + 1 \ldots\ldots$ Here $s_n = \mathrm{n}$. Clearly the sequence (s_n) diverges to ∞. Hence the given series diverges to ∞.
2. Consider the geometric series $1 + r + r^2 + \ldots\ldots + r^n + \ldots.$ Here ,

   $s_n = 1 + r + r^2 + \ldots\ldots + r^{n-1} = \frac{1-r^n}{1-r}$.

   Case (i) $0 < \mathrm{r} < 1$. Then$(r^n) \to 0$

   $\therefore (s_n) \to \frac{1}{1-r}$.

   $\therefore$ The given series converges to the sum $\frac{1}{(1-r)}$

   Case (ii) $\mathrm{r} > 1$. Then $s_n = \frac{r^n - 1}{r-1}$.

Also $(r^n) \rightarrow \infty$ when $r > 1$.

Hence the series diverges to ∞.

Case (iii) $r = 1$. Then the series becomes $1 + 1 + \ldots$.

$\therefore (s_n) = (n)$. which diverges to ∞.

Case (iv) $r = -1$.

Then the series becomes $1 - 1 + 1 - 1 + \ldots\ldots$

$$\therefore s_n = \begin{cases} 0 \ if \ n \ is \ even \\ 1 \ if \ n \ is \ odd \end{cases}.$$

$\therefore (s_n)$ oscillates finitely.

Hence the given series oscillates finitely.

Case (v) $r < -1$.

$\therefore (r_n)$ oscillates infinitely

$\therefore (s_n)$ oscillates infinitely.

Hence the given series oscillates infinitely.

3. Consider the series $1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!} + \ldots\ldots$

Then $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{(n-1)!}$

The sequence $(s_n) \rightarrow e$

$\therefore$ The given series diverges to ∞.

Note 1. Let $\sum a_n$ be a series of positive terms. Then (s_n) is a monotonic increasing sequence. Hence (s_n) converges or diverges to ∞ according as (s_n) is bounded or unbounded. Hence the series $\sum a_n$ converges or diverges to ∞.

Thus a series of positive terms cannot oscillate.

Note 2. Let $\sum a_n$ be a convergent series of positive terms converging to the sum s. Then s is the *l. u. b.* of (s_n). Hence $s_n \leq s$ for all n.

Also given $\epsilon > 0$ there exists m ∈ **N** such that $s \rightarrow \epsilon < s_n$ for all n ≥ m.

Hence $s \rightarrow \epsilon < s_n \leq s$ for all n ≤ m.

Theorem 4.1. Let $\sum a_n$ be a convergent series converging to the sum s. Then $\lim_{n\to\infty} a_n = 0$

Proof. $\lim_{n\to\infty} a_n = \lim\limits_{n\to\infty} (s_n - s_{n-1})$

$$= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$$

$$= s - s = 0.$$

Note 1. The converse of the above theorem is not true. ie., If $\lim a_n = 0$, then $\sum a_n$ need not converge. For example, consider the series $\sum \frac{1}{n}$. Here $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. However the series $\sum \frac{1}{n}$ diverges.

Note 2. If $\lim a_n \neq 0$ then the series $\sum a_n$ is not convergent. If further $\sum a_n$ is a series of positive terms then the series cannot oscillate and hence the series diverges.

Theorem 4.2. Let $\sum a_n$ converge to a and $\sum b_n$ converge to b. Then $\sum(a_n \neq b_n)$ converges to $a \neq b$ and $\sum ka_n$ converges to ka.

Proof. Let $s_n = a_1 + a_2 + \ldots + a_n$ and $t_n = b_1 + b_2 + \ldots + b_n$.

Then $(s_n) \rightarrow a$ and $(t_n) \rightarrow b$.

$\therefore (s_n \neq t_n) \rightarrow a \neq b$

Also $(s_n \neq t_n)$ is the sequence of partial sums of $\sum(a_n \neq b_n)$.

$\therefore \sum(a_n \neq b_n)$ converges to $a \neq b$.

Similarly $\sum ka_n$ converges to ka.

Theorem 4.3 (Cauchy's general principle of convergence)

The series $\sum a_n$ is convergent iff given $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon$ for all $n \geq n_0$ and for all positive integers p.

Proof. Let $\sum a_n$ be a convergent series.

Let $s_n = a_1 + \ldots + a_n$.

$\therefore (s_n)$ is a convergent sequence.

$\therefore (s_n)$ is a Cauchy sequence

$\therefore$ There exists $n_0 \in \mathbf{N}$ such that $|s_{n+p} - s_n| < \epsilon$ for all $n \geq n_0$ and for all $p \in \mathbf{N}$.

$\therefore |a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon$ for all $n \geq n_0$ and for all $p \in \mathbf{N}$.

Conversely if $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon$ for all $n \geq n_0$ and for all $p \in \mathbf{N}$ then (s_n) is a Cauchy sequence in $\mathbf{R}$ and hence (s_n) is convergent.

$\therefore$ The given series converges.

Solved Problems.

1. Apply Cauchy's general principle of convergence to show that the series $\sum(\frac{1}{n})$ is not convergent.

Solution. Let $s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$.

Suppose the series $\sum(\frac{1}{n})$ is convergent.

$\therefore$ By Cauchy's general principle of convergence, given $\epsilon > 0$ there exists $m \in \mathbf{N}$ such that $|s_{n+p} - s_n| < \epsilon$ for all $n \geq m$ and for all $p \in \mathbf{N}$.

$\therefore \left|1 + \frac{1}{2} + \ldots + \frac{1}{n+p} - (1 + \frac{1}{2} + \ldots + \frac{1}{n})\right| <$ for all $n \geq m$ and for all $p \in \mathbf{N}$.

$\therefore \left|\frac{1}{n+1} + \frac{1}{n+2} \ldots + \frac{1}{n+p}\right| < \epsilon$ for all $n \geq m$ and for all $p \in \mathbf{N}$.

In particular if we take $n = m$ and $p = m$ we obtain $\frac{1}{m+1} + \frac{1}{m+2} \dots + \frac{1}{m+m} >$ $\frac{1}{2m} + \cdots + \frac{1}{2m} = \frac{1}{2}$.

$\therefore \frac{1}{2} < \epsilon$ which is a contradiction since $\epsilon > 0$ is arbitray.

$\therefore$ The given series is not convergent.

2. Applying Cauchy's general principle of convergence prove that $1 + \frac{1}{2} + \frac{1}{3} \dots + (-1)^n \frac{1}{n} + \dots$ is convergent.

Solution. Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} \dots + \frac{(-1)^n}{n}$.

$$\therefore \left|s_{n+p} - s_n\right| = \left|\frac{1}{n+1} + \frac{1}{n+2} \dots + \frac{(-1)^{p-1}}{n+p}\right|$$

Now, $\frac{1}{n+1} - \frac{1}{n+2} \dots + \frac{(-1)^{p-1}}{n+p}$

$$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \dots\dots + \begin{cases} \frac{1}{n+p-1} - \frac{1}{n+p} & if\ p\ is\ even \\ \frac{1}{n+p} & if\ p\ is\ odd \end{cases}$$

$$> 0$$

$$\therefore \left|s_{n+p} - s_n\right| = \frac{1}{n+1} - \frac{1}{n+2} + \dots + \frac{(-1)^{p-1}}{n+p}$$

$$= \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \dots\dots$$

$$< \frac{1}{n+1}.$$

$$< \epsilon \text{ provided n } > \left(\frac{1}{\epsilon} - 1\right).$$

$\therefore$ By Cauchy's general principle, the given series is convergent.

Exercises.

1. Show that the series $\sum(\frac{1}{2n})$ converges to the sum 1.
2. If $\sum c_n$ is a convergent series of positive terms then so is $\sum a_n\, c_n$ where (a_n) is a bounded sequence of positive terms.

3. If $\sum d_n$ is a divergent sequence of positive terms then so is $\sum a_n\, d_n$ where (a_n) is a sequence of positive lower bound.
4. Show that $\frac{2}{5}+\frac{4}{5^2}+\frac{2}{5^3}+\frac{4}{5^4}+\frac{2}{5^5}+\frac{4}{5^6}+\ldots\ldots=\frac{7}{12}$ (Hint: Express this series as the sum of two geometric series).
5. Let a and b be two positive real numbers. Show that the series $a+b+a^2+b^2+a^3+b^3$ …………… Converges if both a and $b<1$ and diverges if either $a>1$ or $b>1$.
6. Show that the series $\sum cos(\frac{1}{n})$ is divergent.

   (Hint: Consider the limit of the n th term).

Comparison test

Theorem 4.4 (Comparison test)

(i) Let $\sum c_n$ be a convergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in \mathbf{N}$ such that $a_n \leq c_n$ for all $n \geq m$, then $\sum a_n$ is also convergent.

(ii) Let $\sum d_n$ be a divergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in \mathbf{N}$ such that $a_n \leq d_n$ for all $n \geq m$, then $\sum a_n$ is also divergent.

Proof (i) Since the convergence or divergence of a series is not altered by the removal of a finite number of terms we may assume without loss of generality that $a_n \leq c_n$ for all n.

Let $s_n = c_1 + c_2 + \ldots\ldots + c_n$ and $t_n = a_1 + a_2 + \ldots\ldots + a_n$.

Since $a_n \leq c_n$ we have $t_n \leq s_n$.

Now, Since $\sum c_n$ is convergent, (s_n) is a convergent sequence.

$\therefore$ (s_n) is a bounded sequence.

$\therefore$ There exists a real positive number k such that $s_n \leq k$ for all n.

$\therefore$ $t_n \leq k$ for all $n \geq m$

Hence (t_n) is bounded above.

Also (t_n) is a monotonic increasing sequence.

$\therefore (t_n)$ converges

$\therefore \sum a_n$ converges.

(ii)Let $\sum d_n$ diverge and $a_n \geq d_n$ for all n.

$\therefore t_n \geq s_n$.

Now, (s_n) is diverges to ∞.

$\therefore$ (s_n) is not bounded above.

$\therefore (t_n)$ is not bounded above.

Further (t_n) is monotonic increasing and hence (t_n) diverges to ∞.

$\therefore$ $\sum a_n$ diverges to ∞.

Theorem 4.5.

(i) If $\sum c_n$ converges and if $\lim_{n\to\infty} \frac{a_n}{c_n}$ exists and is finite then $\sum a_n$ also converges.

(ii) If $\sum d_n$ diverges and if $\lim_{n\to\infty} \frac{a_n}{d_n}$ exists and is greater than zero then $\sum a_n$ diverges.

Proof (i) . Let $\lim_{n\to\infty} \frac{a_n}{c_n} = k$.

Let $\epsilon > 0$ be given. Then there exists $n_1 \in \mathbf{N}$ such that $\frac{a_n}{c_n} < k + \epsilon$ for all $n \geq n_1$.

$\therefore$ $a_n < (k+\epsilon)\, c_n$ for all $n \geq n_1$.

Also since $\sum c_n$ is a convergent series, $\sum (k + \epsilon)\, c_n$ is also convergent series.

$\therefore$ By comparision test $\sum a_n$ is convergent.

(ii)Let $\lim_{n\to\infty} \frac{a_n}{d_n} = k > 0$.

Choose $= \frac{1}{2}k$. Then there exists $n_1 \in \mathbf{N}$ such that $k - \frac{1}{2}k < \frac{a_n}{d_n} < k + \frac{1}{2}k$ for all $n \geq n_1$.

$\therefore\ \frac{a_n}{d_n} > \frac{1}{2}k$ for all $n \geq n_1$.

$\therefore\ a_n > \frac{1}{2}k\ d_n$ for all $n \geq n_1$.

Since $\sum d_n$ is a divergent series, $\sum \frac{1}{2}k\ d_n$ is also divergent series.

$\therefore$ By comparision test, $\sum a_n$ diverges.

Theorem 4.6.

(i) Let $\sum c_n$ be a convergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in \mathbf{N}$ such that $\frac{a_{n+1}}{a_n} \leq \frac{c_{n+1}}{c_n}$ for all $n \geq m$, then Let $\sum a_n$ is convergent.

(ii) Let $\sum d_n$ be a divergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in \mathbf{N}$ such that $\frac{a_{n+1}}{a_n} \leq \frac{d_{n+1}}{d_n}$ for all $n \geq m$, then Let $\sum a_n$ is divergent.

Proof. (i) $\frac{a_{n+1}}{c_{n+1}} \leq \frac{a_n}{c_n} \left(\because \frac{a_{n+1}}{a_n} \leq \frac{c_{n+1}}{c_n}\right)$

$\therefore\ \frac{a_n}{c_n}$ is a monotonic decreasing sequence.

$\therefore\ \frac{a_n}{c_n} \leq k$ for all n where $k = \frac{a_1}{c_1}$.

$\therefore\ a_n \leq kc_n$ for all $n \in \mathbf{N}$.

Now, $\sum c_n$ is convergent. Hence $\sum kc_n$ is also a convergent series of positive terms.

$\therefore\ \sum a_n$ is also convergent

(ii)Proof is similar to that of (i).

Note 1. Theorem 4.5 and 4.6 are alternative forms of the comparison test mentioned in theorem 4.4 and these forms of the comparison test are often easier to work with.

Note 2. The comparison test can be used only if we already have a large number of series whose convergence or divergence are known. We know that a geometric series

$\sum r^n$ converges if $0 \le r < 1$ and diverges if $r \ge 1$. In the following theorem we give another family of series whose behavior is known.

Theorem 4.7. The **harmonic series** $\sum -\frac{1}{n^p}$ converges if $p > 1$ and if $p \le 1$.

Proof.

Case (i) Let $p = 1$. Then the series becomes $\sum(\frac{1}{n})$ which diverges.

Case (ii) Let $p < 1$. Then $n^p < n$ for all n.

$\therefore \frac{1}{n^p} > \frac{1}{n}$.

$\therefore$ By comparison test $\sum \frac{1}{n^p}$ diverges.

Case (iii) Let $p > 1$.

Let $S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots.. + \frac{1}{n^p}$.

Then $S_{2^{n+1}-1} = 1 + \frac{1}{2^p} + \ldots.. + \frac{1}{(2^{n+1}-1)^p}$

$$= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots. + \frac{1}{(2^n)^p} + \frac{1}{(2^{n+1})^p} + \ldots$$

$$+ \frac{1}{(2^{n+1}-1)^p}$$

$$< 1 + 2\left(\frac{1}{2^p}\right) + 4\left(\frac{1}{4^p}\right) + \ldots + 2^n\left(\frac{1}{(2^n)^p}\right)$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2p-2}} + \ldots. + \frac{1}{(2^{(p-1)})^n}$$

$$\therefore \; S_{2^{n+1}-1} < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \ldots.. + \left(\frac{1}{2^{p-1}}\right)^n.$$

Now, since $p > 1$, $p - 1 > 0$. Hence $\frac{1}{2^{p-1}} \le 1$.

$$\therefore \; 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \ldots.. + \left(\frac{1}{2^{p-1}}\right)^n < \frac{1}{1 - \frac{1}{2^{p-1}}} = k \text{ (say)}.$$

$$\therefore \; S_{2^{n+1}-1} < k.$$

Now let n be any positive integer. Choose $m \in \mathbf{N}$ such that $n \leq 2^{m+1} - 1$. Since (s_n) is a monotonic increasing sequence , $s_n \leq S_{2^{m+1}-1}$.

Hence $s_n < k$ for all n.

Thus (s_n) is a monotonic increasing sequence and is bounded above.

$\therefore$ (s_n) is convergent.

$\therefore$ $\sum \frac{1}{n^p}$ is convergent.

Solved problems.

1. Discuss the convergence of the series $\sum \frac{1}{\sqrt{(n^3+1)}}$

Solution. $\frac{1}{\sqrt{(n^3+1)}} < \frac{1}{n^{3/2}}$.

Also $\sum \frac{1}{n^{3/2}}$ is convergent

$\therefore$ By comparision test, $\sum \frac{1}{\sqrt{(n^3+1)}}$ is convergent.

2. Discuss the convergence of the series $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n^p}$.

Solution. $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^p}$

$$= \frac{n+1-n}{n^p(\sqrt{n+1}+\sqrt{n})}$$

$$= \frac{1}{n^p(\sqrt{n+1}+\sqrt{n})}$$

Now, let $b_n = \frac{1}{n^{p+\frac{1}{2}}}$.

$$\therefore \sum_{n\to\infty} \frac{a_n}{b_n} = \sum_{n\to\infty} \frac{n^{p+\frac{1}{2}}}{n^p(\sqrt{n+1}+\sqrt{n})}$$

$$= \sum_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1}$$

$$= \frac{1}{2}.$$

Also $\sum_{n\to\infty} b_n$ is convergent if $p+\frac{1}{2}>1$ and divergent if $p+\frac{1}{2}<1$

$\therefore\ \sum_{n\to\infty} a_n$ is convergent if $p>\frac{1}{2}$ and divergent if $p<\frac{1}{2}$.

3. Discuss the convergence of the series $\sum \frac{1^2+2^2+\cdots+n^2}{n^4+1}$.

Solution. Let $a_n = \frac{1^2+2^2+\cdots+n^2}{n^4+1}$

$$= \frac{n(n+1)(2n+1)}{6(n^4+1)}.$$

Now , let $b_n = \frac{1}{n}$.

$$\therefore\ \sum_{n\to\infty} \frac{a_n}{b_n} = \sum_{n\to\infty} \frac{n^2(n+1)(2n+1)}{6(n^4+1)}$$

$$= \sum_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)(2+\frac{1}{n})}{6(1+\frac{1}{n^4})}$$

Also $\sum b_n$ is divergent

$\therefore\ \sum a_n$ is divergent

4. Discuss the convergence of the series $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\ldots..$

Solution. Let $a_n = \frac{n^n}{(n+1)^{n+1}}$.

Let $b_n = \frac{1}{n}$.

$$\therefore\ \sum_{n\to\infty} \frac{a_n}{b_n} = \sum_{n\to\infty} \frac{n^{n+1}}{(n+1)^{n+1}}$$

$$= \sum_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}$$

$$= \frac{1}{\epsilon} > 0.$$

Also $\sum_{n\to\infty} b_n$ is divergent.

$\therefore\ \sum_{n\to\infty} a_n$ is divergent

5. Discuss the convergence of the series $\sum_3^\infty (\log\log \mathrm{n})^{-\log \mathrm{n}}$.

Solution. Let $a_n = (\log\log \mathrm{n})^{-\log \mathrm{n}}$

$\therefore\ a_n = n^{-\theta n}$ where $\theta n = \log\,(\log\log \mathrm{n})$.

Since $\sum_{n\to\infty} \log\ \log\log \mathrm{n} = \infty$, there exists $m \in \mathbf{N}$

such that $\theta n \geq 2$ for all $n \geq m$.

$\therefore\ n^{-\theta n} \leq n^{-2}$ for all $n \geq m$.

$\therefore a_n = n^{-2}$ for all $n \geq m$.

Also $\sum n^{-2}$ is convergent.

$\therefore$ By comparison test the given series is convergent.

6. Show that $\sum \frac{1}{4n^2-1} = \frac{1}{2}$.

Solution. Let $a_n = \frac{1}{4n^2-1}$.

Clearly $a_n < \frac{1}{n^2}$.

Also $\sum \frac{1}{n^2}$ is convergent

$\therefore$ By comparison test, the given series converges

Now, $a_n = \frac{1}{4n^2-1} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$.

$\therefore\ s_n = a_1 + a_2 + \ldots.. + a_n$

$$= \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right] .$$

$\therefore\ \sum_{n\to\infty} s_n = \frac{1}{2}$.

$\therefore \quad \sum \frac{1}{4n^2-1} = \frac{1}{2}$.

Exercises. Discuss the convergence of the following series whose n th terms are given below.

(1) $\frac{5+n}{3+n^2}$ (2) $\frac{1}{n\sqrt{(n^2+1)}}$ (3) $\frac{n}{(n^2+1)^{2/3}}$

(4) $\frac{n}{(n^2+1)^{3/2}}$ (5) $\frac{1}{n-\sqrt{n}}$ (6) $\frac{n(n+1)}{(n+2)(n+3)(n+4)}$

(7) $\frac{1}{a+nx}$ (8) $\frac{(n+1)^2}{n^k+(n+2)^k}$ (9) $\frac{n+1}{n^p}$

(10) $n\sqrt{(n^2+1)} - \sqrt{(n^2}$ (11) $\frac{(n+a)(n+b)}{n(n+1)(n+2)}$ (12) $\frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$

(13) $\frac{1}{n^{\log n}}$

(14) Show that if $\sum a_n$ is convergent then $\sum a_{n^2}$.

$\sum \frac{a_n}{1+a_n}$ and $\sum \frac{a_n}{1+n^2 a_n}$ are also convergent.

Answers: 1.D, 2.C, 3.D, 4.C, 5.D, 6.D, 7.D, 8.C, 9.C if p > 2, 10.C,11.D,12.D,13.C.

Kummer's test

Theorem 4.8 (Kummer's test)

Let $\sum a_n$ be a given series of positive terms and $\sum \frac{1}{d_n}$ be a series of a positive terms diverging to ∞. Then

(i) $\sum a_n$ converges if $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) > 0$ and

(ii) $\sum a_n$ diverges if $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) < 0$.

Proof. (i) Let $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) = l > 0$.

We distinguish two cases.

Case (i) l is finite.

Then given $\epsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$l - \epsilon < d_n \frac{a_n}{a_{n+1}} - d_{n+1} < l + \epsilon \text{ for all } n \geq m$$

$\therefore\ d_n a_n - d_{n+1}a_{n+1} > (l - \epsilon)\ a_{n+1}$ for all $n \geq m$.

Taking $\epsilon = \frac{1}{2}\ l$, we get

$d_n a_n - d_{n+1}a_{n+1} > \frac{1}{2}\ l\ a_{n+1}$ for all $n > m$.

Now , let $n \geq m$

$\therefore\ d_m a_m - d_{m+1}a_{m+1} > \frac{1}{2}\ l\ a_{m+1}$

$d_{m+1}a_{m+1} - d_{m+2}a_{m+2} > \frac{1}{2}\ l\ a_{m+2}$

…………………………………..

…………………………………..

$d_{n-1}a_{n-1} - d_n a_n > \frac{1}{2}\ l\ a_n$.

Adding, we get

$d_m a_m - d_n a_n > \frac{1}{2}\ l(a_{m+1} + \cdots + a_n)$

$\therefore d_m a_m - d_n a_n > \frac{1}{2}\ l\ (s_n - s_m)$ where $s_n = a_1 + a_2 + \ldots + a_n$.

$\therefore d_m a_m > \frac{1}{2}\ l\ (s_n - s_m)$.

$\therefore\ s_n < \frac{2d_m a_m + l - s_m}{l}$ which is independent of n.

$\therefore$ The sequence (s_n) of partial sum is bounded.

$\therefore\ \sum a_n$ is convergent.

Case (ii) $l = \infty$.

Then given real number $k > 0$ there exists a positive integer m such that

$d_n\left(\frac{a_n}{a_{n+1}}\right) - d_{n+1} > k$ for all $n \geq m$.

$\therefore d_n a_n - d_{n+1}a_{n+1} > k\ a_{n+1}$ for all $n \geq m$.

Now, let $n \geq m$. Writing the above inequality for m , $m+1, \ldots..(n-1)$ and adding we get

$$d_m a_m - d_n a_n > k\,(a_{m+1} + \cdots + a_n)$$

$$= k(s_n - s_m).$$

$\therefore\ d_m a_m > k\,(s_n - s_m).$

$\therefore\ s_n < \frac{d_m a_m}{k} + s_m.$

$\therefore$ The sequence (s_n) is bounded and hence $\sum a_n$ is convergent.

$(ii)\ \lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) = l < 0.$

Suppose l is finite.

Choose $\epsilon > 0$ such that $l + \epsilon < 0$.

Then there exists $m \in \mathbf{N}$ such that

$l + \epsilon < d_n \frac{a_n}{a_{n+1}} - d_{n+1} < l + \epsilon < 0$ for all $n \geq m$.

$\therefore\ d_n a_n < d_{n+1} a_{n+1}$ for all $n \geq m$.

Now let $n \geq m$

$\therefore\ d_m a_m < d_{m+1} a_{m+1}$

……………………………….

……………………………….

$d_{n-1} a_{n-1} < d_n a_n$

$\therefore\ d_m a_m < d_n a_n.$

$\therefore\ a_n > \frac{d_m a_m}{d_n}.$

Also, by hypothesis $\sum \frac{1}{d_n}$ is divergent.

Hence $\sum_{n=1}^{\infty} \frac{d_m a_m}{d_n}$ is divergent.

∴ By comparison test $\sum a_n$ is divergent.

The proof is similar if $l = -\infty$.

Note1. The above test fails if $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right)$ =0.

Note2. The divergence of $\sum(1/d_n)$ has not been used in the proof of (i).

Corollary 1.(D' Alembert's ratio test)

Let $\sum a_n$ be a series of positive terms. Then $\sum a_n$ converges if $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$ and diverges if $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

Proof. The series $1 + 1 + 1 + \ldots$. is divergent

∴ We can put $d_n = 1$ in Kummer's test.

Then $d_n \frac{a_n}{a_{n+1}} - d_{n+1} = \frac{a_n}{a_{n+1}} = 1$

∴ $\sum a_n$ converges if $\lim\limits_{n\to\infty}\left(\frac{a_n}{a_{n+1}} - 1\right) > 0$

∴ $\sum a_n$ converges if $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$.

Similarly $\sum a_n$ diverges if $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

Corollary 2. (Raabe's test)

Let $\sum a_n$ be a series of positive terms . Then $\sum a_n$ converges if $\lim\limits_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$ and diverges if $\lim\limits_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$.

Proof. The series $\sum \frac{1}{n}$ is divergent.

∴ We can put $d_n = n$ in Kummer's test.

Then $d_n \frac{a_n}{a_{n+1}} - d_{n+1} = n\frac{a_n}{a_{n+1}} - (n+1)$

$$= n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1$$

$\therefore \quad \sum a_n$ converges if $\lim\limits_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$ and diverges if $\lim\limits_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$.

Solved problems.

1. Test the convergence of the series $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \ldots.$

Solution. Let $a_n = \frac{1.2.3\ldots.n}{3.5.7\ldots..(2n+1)}$.

$\therefore \quad \frac{a_n}{a_{n+1}} = \frac{2n+3}{n+1} = \frac{2+3/n}{1+1/n}$.

$\therefore \lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = 2 > 1.$

$\therefore$ By D' Alembert's ration test $\sum a_n$ is convergent.

2. Test the convergence of $\sum \frac{n^n}{n!}$.

$\therefore \frac{a_n}{a_{n+1}} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$\therefore \lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{\epsilon} < 1$

$\therefore \sum a_n$ is divergent.

3. Test the convergence of the series $\sum \frac{2^n n!}{n^n}$.

Solution. Let $a_n = \frac{2^n n!}{n^n}$.

$\therefore \frac{a_n}{a_{n+1}} = \frac{(n+1)^{n+1}}{(n+1)n^n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)^n$.

$\therefore \lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{e}{2} > 1.$

$\therefore$ By ratio test the series converges.

4. Test the convergence of the series $\sum \frac{3^n n!}{n^n}$.

Solution. As in the above problem, we find that $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{e}{2} < 1$

$\therefore$ By ratio test the series diverges.

5. Test the convergence of the series $\sum \sqrt{\frac{n}{n+1}}\, x^n$ where x is any positive real number.

Solution. Since x is positive the given series is a series of positive terms.

Now, $\frac{a_n}{a_{n+1}} = \sqrt{\frac{n(n+2)}{(n+1)^n}}\left(\frac{1}{x}\right)$

$= \frac{\sqrt{(1+\frac{2}{n})}}{1+1/n}\left(\frac{1}{x}\right).$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x}$.

$\therefore$ By ratio test $\sum a_n$ converge if $x < 1$ and diverges if $x > 1$.

If $x = 1$ the test fails.

When $x = 1$, $a_n = \sqrt{\frac{n}{n+1}} = \frac{1}{\sqrt{(1+1/n)}}.$

$\therefore \lim_{n\to\infty} a_n = 1.$

$\therefore$ The series diverges .

6. Test the convergence of the series $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots..$ where x is any positive real number.

Solution. Since x is a positive real number, the given series is a series of positive terms.

Let $a_n = \frac{x^{2n-2}}{2n-2}$, (n > 1).

$\therefore \frac{a_n}{a_{n+1}} = \frac{2n}{2n-2}\left(\frac{1}{x^2}\right).$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x^2}.$

$\therefore$ The series test , the series converges if $x^2 < 1$ and diverges if $x^2 > 1$.

$\therefore$ The series converges if $x < 1$ and diverges if $x > 1$. If $x = 1$ the test fails.

When $x = 1 \quad a_n = \frac{1}{2n-2}$.

By comparing with the series $\sum(\frac{1}{n})$ we see that the series diverges.

7. Test the converges of the series $\sum \frac{n^2+1}{5^n}$.

Solution. $\frac{a_n}{a_{n+1}} = \frac{5(n^2+1)}{(n+1)^2+1}$

$$= \frac{5(n^2+1)}{n^2+2n+2}$$

$$= \frac{5\left(1+\frac{1}{n^2}\right)}{n^2+2n+2}.$$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 5.$

$\therefore$ By ratio test the series converges.

8. Test the convergence of the series $\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\ldots..$

Solution. Let $a_n = \frac{1}{2^n} + \frac{1}{3^n}$

$$= \frac{2^n + 3^n}{2^n 3^n}.$$

$\therefore \frac{a_n}{a_{n+1}} = \frac{6(2^n + 3^n)}{2^{n+1} + 3^{n+1}}.$

$$= \frac{2[2^n + (\frac{2}{3})^n]}{[1+\left(\frac{2}{3}\right)^{n+1}]}$$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 2.$

$\therefore$ By ratio test the given series converges.

9. Test the convergence of the series $\sum \frac{x^n}{n}$.

Solution. Let $a_n = \frac{x^n}{n}$.

$\therefore \frac{a_n}{a_{n+1}} = \frac{n+1}{n}\left(\frac{1}{x}\right).$

$$= \left(1 + \frac{1}{n}\right)\left(\frac{1}{x}\right).$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x}.$$

$\therefore$ The series converges if $x < 1$ and diverges if $x > 1$.

If $x = 1$, the series converges if $x < 1$ and diverges if $x > 1$.

If $x = 1$, the series becomes $\sum \frac{1}{n}$ which is divergent.

10. Test the convergence of the series $\frac{n^p}{n!}$ $(p > 0)$.

Solution. Let $a_n = \frac{n^p}{n!}$.

$$\therefore \frac{a_n}{a_{n+1}} = \frac{n^p(n+1)}{(n+1)^p}$$

$$= \frac{n+1}{(1+1/n)^p}.$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \infty .$$

$\therefore$ By ratio test $\sum a_n$ is convergent.

11. Test the convergence of the series $\frac{1}{3}x + \frac{1}{3}\,\frac{2}{5}x^2 + \frac{1}{3}\,\frac{2}{5}\,\frac{3}{7}x^3 + \ldots\ldots$

Solution. Let $a_n = \frac{1.2.3\ldots\ldots\ldots n}{3.5.7(2n+1)}x^n$.

$$\therefore \frac{a_n}{a_{n+1}} = \frac{2n+3}{n+1}\left(\frac{1}{x}\right).$$

$$= \frac{2+3/n}{1+1/n}\left(\frac{1}{x}\right).$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{2}{x}.$$

$\therefore$ By ratio test the series converges if $\frac{2}{x} > 1$.

$\therefore$ The series converge if $x < 2$ and diverges if $x > 2$.

If $x = 2$, the ratio test fails.

In this case $\frac{a_n}{a_{n+1}} = \frac{2n+3}{2n+2}$

$\therefore \frac{a_n}{a_{n+1}} - 1 = \frac{1}{2n+2}$.

$\therefore n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n+2} = \frac{1}{2+2/n}$.

$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{1}{2}$.

$\therefore$ By Raabe's test the series diverges.

Exercises. Test the convergence of the following series.

(1) $1 + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \ldots..$

(2) $\sum \frac{x^n}{\sqrt{(2n+3)}}$

(3) $1 + a + \frac{a(a+1)}{2!} + \frac{(a+1)(a+2)}{3!} + \ldots.$

(4) $\frac{1}{x}x + \frac{2!}{3.5}x^2 + \frac{3!}{3.5.7}x^3 + \ldots\ldots$

(5) $\sum \frac{\sqrt{n}}{n+1} x^n$

(6) $\sum \frac{x^n}{(2n-1)2n}$

(7) $\sum \frac{n+1}{n^3} x^n$

(8) $\frac{x}{1} + \frac{1}{2} \cdot \frac{x^2}{3} + \frac{1}{2} \cdot \frac{x^2}{3} \cdot \frac{x^3}{5} + \ldots.$

(9) $1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \ldots..$

Answers : 1.C if $\beta > \alpha$, 2.C if $0 \le x < 1$, 3.C if $a \le 0$, 4.C if $x < 2$, 5. C if $x > 1$, 6. C if $x \le 1$, 7. C if $0 \le x \le 1$, 8. C if $x^2 \le 1$, 9. D.

Root test and condensation test

Theorem 4.10 (Cauchy's root test)

Let $\sum a_n$ be a series of positive terms. Then $\sum a_n$ is convergent if $\lim_{n\to\infty} a_n{}^{1/n} < 1$ and divergent if $\lim_{n\to\infty} a_n{}^{1/n} > 1$

Proof. Case(i) let $\lim_{n\to\infty} a_n{}^{1/n} = l < 1$.

Choose $\epsilon > 0$ such that $l + \epsilon < 1$. Then there exists $m \in$ N such that $a_{n^{1/n}} < l + \epsilon$ for all $n \geq m$

$\therefore a_n < (l + \epsilon)^n$ for all $n \geq m$.

Now since $l + \epsilon < 1$, $\sum (l + \epsilon)^n$ is convergent.

$\therefore$ By comparison test $\sum a_n$ is convergent.

Case (ii) Let $\lim_{n\to\infty} a_{n^{1/n}} = l > 1$.

Choose $\epsilon > 0$ such that $l - \epsilon > 1$.

Then there exists $m \in \boldsymbol{N}$ such that

$a_{n^{1/n}} > l - \epsilon$ for all $n \geq m$.

$\therefore a_n > (l - \epsilon)^n$ for all $n \geq m$.

Now, since $l - \epsilon$, $\sum (l - \epsilon)^n$ is divergent

$\therefore$ By comparison test, $\sum a_n$ is divergent.

Note. The following is a more general form of Cauchy's root test. Let $\sum a_n$ be a series of positive terms. Then $\sum a_n$ is convergent if *lim sup* $a_{n^{1/n}} < 1$ and divergent if *lim sup* $a_{n^{1/n}} > 1$.

Theorem 4.11. (Cauchy's condensation test)

Let $a_1 + a_2 + a_3 + \ldots\ldots + a_n + \ldots..$ ....(1)

Be a series of positive terms and whose terms are monotonic decreasing. Then this series converges or diverges according as the series

$ga_g + g^2 a_{g^2} + \ldots. + g^n a_{g^n} + \ldots..$ ....(2)

converges or diverges where g is any positive integer > 1.

Proof. Let $s_1 = a_1 + a_2 + a_3 + \ldots\ldots + a_n$ and

$$t_1 = ga_g + g^2 \; a_{g^2} + \ldots. + g^n \; a_{g^n}$$

Then $s_{g^n} = (a_1 + a_2 + a_3 + \ldots\ldots + a_g) + (a_{g+1} + a_{g+2} + \ldots\ldots + a_{g^2}) +$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$+ (a_{g^{n-1}+1} + a_{g^{n-1}+2} + \ldots. + a_{g^n})$

$\leq ga_1 + (g^2 - g)\, a_g + \ldots.. + (g^n - g^{n-1})\ a_{g^{n-1}}$

($\because$ The terms of the series are monotonic decreasing).

$= ga_1 + (g - 1)\, a_g + \ldots.. + g^{n-1}(g - 1)\ a_{g^{n-1}}$

$= ga_1 + (g - 1)\,(ga_g + g^2\ a_{g^2} + \ldots. + g^{n-1}\ a_{g^{n-1}})$

$= ga_1 + (g - 1)\, t_{n-1}.$

$\therefore\ s_{g^n} \leq ga_1 + (g - 1)\ t_{n-1}.$

$\therefore$ If the series (2) converges, then (1) converges.

Now, $s_{g^n} \geq ga_g + (g^2 - g)\ a_{g^2} + \ldots\ldots + (g^n - g^{n-1})\ a_{g^n}$

$= ga_g + \frac{g-1}{g}\,(g^2 a_{g2} + \ldots + g^n a_{gn})$

$= ga_g + \frac{g-1}{g}(t_n - ga_g)$

$= a_g + \frac{g-1}{g}\ t_n.$

$\therefore$ If the series (2) diverges, then (1) diverges.

Solved problem.

1. Test the convergence of $\sum \frac{1}{(\log n)^n}$

Solution. Let $a_n = \frac{1}{(\log n)^n}$

$\therefore \sqrt[n]{a_n} = \frac{1}{\log n}$

$\therefore \lim_{n\to\infty} \sqrt[n]{a_n} = 0 < 1.$

$\therefore$ By Cauchy's root test $\sum \frac{1}{(\log n)^n}$ converges.

2. Test the convergence of $\sum \left(1+\frac{1}{n}\right)^{-n^2}$.

Solution. Let $a_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$\therefore \sqrt[n]{a_n} = \left(1+\frac{1}{n}\right)^{-n}$

$\therefore \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{e}$

$\therefore$ By Cauchy's root test the series converges.

3. Prove that the series $\sum e^{-\sqrt{n}} x^n$ converges if $0 < x < 1$ and diverge if $x > 1$

Solution. Let $a_n = e^{-\sqrt{n}} x^n$

$\therefore \ a_n{}^{1/n} = e^{-1/\sqrt{n}} x.$

$\therefore \lim_{n\to\infty} \ a_n{}^{1/n} = x.$

$\therefore$ By Cauchy's root test the given series converges if $0 < x < 1$ and diverges if $x > 1$.

4. Test the convergence of $\sum \frac{n^3+a}{2^n+a}$.

Solution. Let $a_n = \frac{n^3+a}{2^n+a}$, $b_n = \frac{n^3}{2^n}$

$$\therefore \frac{a_n}{b_n} = \left(\frac{n^3+a}{2^n+a}\right)\left(\frac{2^n}{n^3}\right) = \left(\frac{n^3+a}{n^3}\right)\left(\frac{2^n}{2^n+a}\right)$$

$$= \left(1+\frac{a}{n^3}\right)\left(\frac{1}{1+(\frac{a}{2^n})}\right).$$

$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = 1.$

$\therefore$ By comparison test, the given series is convergent or divergent according as $\sum \frac{n^3}{2^n}$ is convergent or divergent.

Now , $b_{n^{1/n}} = \left(\frac{n^2}{2^n}\right)^{1/n} = \frac{n^{3/n}}{2}$.

$\lim_{n\to\infty} n^{3/n} = 1$

Also $\lim_{n\to\infty} b_{n^{1/n}} = \frac{1}{2}$

$\therefore$ $\sum b_n$ is convergent

$\therefore$ $\sum a_n$ is convergent.

5. Test the convergence of $\sum \frac{1}{n \log n}$.

Solution. By Cauchy's condensation test, $\sum \frac{1}{n \log n}$ converges or diverges with the series

$$\sum \frac{2^n}{2^n \log 2^n} = \sum \frac{1}{n \log 2} = \frac{1}{\log 2} \sum \frac{1}{n}.$$

Now the series $\sum \frac{1}{n}$ diverges.

$\therefore$ The given series diverges.

6. Test the convergence of the series $\sum \frac{1}{n(\log n)^p}$.

Solution. The given series converges or diverges with the series $\sum \frac{2^n}{2^n (\log 2^n)^p}$.

$$= \sum \frac{1}{(\log 2)^p n^p} = \frac{1}{(\log 2)^p} \sum \frac{1}{n^p}.$$

The series $\sum \frac{1}{np}$ converges if p $>$ 1 and diverges if p $\leq$ 1.

$\therefore$ The given series converges if p $>$ 1 and diverges p $\leq$ 1.

7. Test the convergence of the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots.$

Solution. We have

$$a_{n^{1/n}} = \begin{cases} \left(\frac{1}{3^{n/2}}\right)^{1/n} & if\ n\ is\ even \\ \left(\frac{1}{2^{(n+1)/2}}\right)^{1/n} & if\ n\ is\ odd \end{cases}$$

$$\begin{cases} \frac{1}{\sqrt{3}} \; if\ n\ is\ even \\ \frac{2}{2^{\frac{1}{2}}(1+\frac{1}{n})} \; if\ n\ is\ odd. \end{cases}$$

Now, the sequence $\left(2^{\frac{1}{2}}(1+\frac{1}{n})\right)$ converges to $(1/\sqrt{2})$ as $n \to \infty$.

$\therefore (1/\sqrt{2})$ and $(1/\sqrt{3})$ are the only limit points of the given sequences.

$\mathit{lim\ sup}\ \ a_{n^{1/n}} = (1/\sqrt{2}) < 1.$

$\therefore$ By Cauchy's root test the given series is convergent.

Note. In this case the limit of $a_{n^{1/n}}$ does not exists since $\mathit{lim\ inf}\ \ a_{n^{1/n}} \neq \mathit{lim\ sup}\ \ a_{n^{1/n}}$

Exercises. Test the convergence of the following.

(1) $\left(1-\frac{1}{n}\right)^{n^2}$

(2) $\left(\frac{n}{n+1}\right)^{n^2}$

(3) $\sum \frac{1}{n^n\sqrt{n}}$

(4) $\sum 2^{-n+(-1)^n}$

(5) $\sum e^{-n^2}$

(6) $\sum \frac{n^3}{3^n}$

(7) $\sum \frac{1}{np^n}$

(8) $\sum e^{\sqrt{n}} x^n$

(9) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{(\log\ n)}}$

(10) $\frac{1}{2} + 1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^6}$ ....

Answers : 1.C, 2. C, 3.C, 4. C, 5. C, 6. C, 7. C if $p > 0$, 8. C, 9. D, 10. C.

UNIT V: SUMMATION OF SERIES

Summation of series using Binomial, Exponential and Logarithmic series.

BINOMIAL SERIES

When n is a positive integer $(x + a)^n$ can be expanded as $(x+a)^n = x^n + {}_nC_1 . x^{n-1}a + {}_nC_2 . x^{n-2}a^2 + \ldots + {}_nC_r . x^{n-r} . a^r + \ldots + a_n$.. This is known as the binomial theorem for the positive integer n. When n is a rational number $(1 + x)^n$ can be expanded as an infinite series when $-1 < x < 1$ (i.e) $|x| < 1$ and it is given by

$$(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (1)$$

This is known as binomial series for $(1+ x)^n$ where n is a rational number.

General term

The (r + 1) th term in the expansion is often denoted by

U_{r+1} or T_{r+1} . $U_{r+1} = {}_nC_r \, x^{n-r} \, a^r$

We may obtain any particular term by giving r particular values. Thus the first term is obtained by writing *r* = 0, the second by writing r = 1 and so on . So the (r +1) th term is called the general term.

Thus we get $(x+a)^n = \sum_{r=0}^{n} {}_nC_r x^{n-r} a^r$

Note:-

(1) The expansion contains (n + 1) terms.

(2) The numbers ${}_nC_0, {}_nC_1 \ldots . {}_nC_r \ldots .. {}_nC_n$ are called the Binomial Coefficients. They are sometimes written as C_0 , C_1 , C_n.These binomial coefficients are all integers since ${}_nC_r$ is the number of combinations of *n* things taken *r* at a time.

(3) Since $C_0 = C_n$, $C_1 = C_{n-1}$, …… $C_r = C_{n-r}$, the coefficients of terms equidistant from the beginning and the end of the expansion are equal.

Summation of various series involving Binomial Coefficients

It is convenient to write the Binomial theorem in the form

$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_rx^r + \ldots C_nx^n$.

We can seen in the expansion that the coefficients of terms which are equidistant from the beginning and the end are equal.

∴ $C_0 = C_n = 1$, $C_1 = C_{n-1} = n$ …. and in general.

$$C_r = C_{n-r} = \frac{n!}{r!(n-r)!}.$$

Some important particular cases of the Binomial expansion.

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots..$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots..$$

$$(1-x)^{-3} = \frac{1}{2}\{1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \ldots.\}$$

$$(1-x)^{-4} = \frac{1}{6}\{1.2.3 + 2.3.4x + 3.4.5x^2 + 4.5.6x^3 + \ldots\}$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$$

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots\ldots$$

$$(1-x)^{-1/3} = 1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \ldots\ldots$$

Application of the Binomial theorem to the summation of series.

We have proved when $|x| < 1$, for all values of n

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$$

Solved problems

Example 1. Find the sum to infinity of the series $1 + \frac{3}{4} + \frac{3}{4} \cdot \frac{5}{8} + \frac{3}{4} \cdot \frac{5}{8} + \frac{7}{12} + \ldots$

Solution.

The factors in the numerators form an A.P with common difference 2: we therefore divide each of these by 2.

Each of the factors in the denominator has 4 for a factor; removing 4 from each will leave a factorial . Hence we have

$$1 + \frac{\frac{3}{2}}{1} \cdot \frac{2}{4} + \frac{\frac{3}{2} \cdot \frac{5}{2}}{1.\ 2} \cdot \left(\frac{2}{4}\right)^2 + \frac{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}}{1.\ 2.\ 3} \cdot \left(\frac{2}{4}\right)^3 + \ldots\ldots$$

i.e., $1 + \frac{\frac{3}{2}}{1!} \cdot \frac{1}{2} + \frac{\frac{3}{2} \cdot \frac{5}{2}}{2!} \cdot \left(\frac{2}{4}\right)^2 + \frac{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}}{3!} \cdot \left(\frac{2}{4}\right)^3 + \ldots\ldots$

Put $n = \frac{3}{2}$ and $x = \frac{1}{2}$.

Then the series becomes

$$1 + \frac{n}{1!} x + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots\ldots$$

$$= (1 - x)^{-n}$$

$$= (1 - \tfrac{1}{2})^{-3/2}$$

$$= 2\sqrt{2}.$$

Example 2. Sum the series to infinity $\frac{1\ .\ 4}{5\ .\ 10} - \frac{1\ .\ 4\ .\ 7}{5\ .\ 10\ .\ 15} + \frac{1\ .\ 4\ .\ 7\ .\ 10}{5\ .\ 10\ .\ 15\ .\ 20} \ldots..$

Solution.

The numerators form an A.P . with 3 as common difference and the denominators are factorials, each of whose factors has been multiplied by 5.

∴ The series can be written as

$$S = \frac{\frac{1}{3}\cdot\frac{4}{3}}{1\ .\ 2}\cdot\left(-\frac{3}{5}\right)^2 + \frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}}{1\ .\ 2\ .\ 3}\cdot\left(-\frac{3}{5}\right)^3 + \frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdot\frac{10}{3}}{1\ .\ 2\ .\ 3\ .\ 4}\cdot\left(-\frac{3}{5}\right)^4 + \ldots.$$

Put $n = \frac{1}{3}$ and $x = -\frac{3}{5}$.

$$\therefore\ S = \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \frac{n(n+1)(n+2)(n+3)}{4!}x^4\ldots..$$

$$= 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3$$

$$= (1-x)^{-n} - 1 - nx$$

$$= (1+\tfrac{3}{5})^{-1/3} - 1 + \frac{1}{3}\cdot\frac{3}{5} = \frac{1}{2}\,(5)^{1/3} - \frac{4}{5}.$$

Example 3. Sum the series to infinity. $\frac{15}{16} + \frac{15.21}{16.24} + \frac{15.21.27}{16.24.32} + \ldots\ldots$

Solution.

The factors in the numerator form an A.P. with common difference 6 and those of the denominator an A.P with common difference 8.

Let S be the sum of the series.

$$\text{Then } S = \frac{\frac{15}{6}}{2}.\left(\frac{6}{8}\right) + \frac{\frac{15}{6}.\frac{21}{6}}{2\ .3}.\left(\frac{6}{8}\right)^2 + \frac{\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{2.3.4}.\left(\frac{6}{8}\right)^3 + \ldots..$$

The factors of the denominators do not begin with 1. Hence one additional factor , namely unity, has to be introduced into the denominator of each coefficient. The number of factors in the numerator is to be the same as that of the factors in the denominator. So we have to introduce an additional factor in the numerator also, which factor is clearly $\frac{9}{6}$.

$$\therefore\ \frac{9}{6}S = \frac{\frac{9}{6}.\frac{15}{6}}{1.2}\left(\frac{6}{8}\right) + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}}{1.2.3}\left(\frac{6}{8}\right)^2 + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{1.2.3.4}\left(\frac{6}{8}\right)^3 + \ldots$$

Since the index of x in every term must be the same as the number of factors in the numerator or denominator of the coefficient, we have

$$S.\frac{9}{6}.\frac{6}{8} = \frac{\frac{9}{6}.\frac{15}{6}}{2!}\left(\frac{6}{8}\right)^2 + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}}{3!}\left(\frac{6}{8}\right)^3 + \ldots$$

Put $\frac{9}{6} = \text{n}$ and $x = \frac{6}{8}$.

$$\therefore \frac{9}{6}\text{S} = \frac{\text{n(n+1)}}{2!}x^2 + \frac{\text{n(n+1)(n+2)}}{3!}x^3 + \ldots\ldots$$

$$= 1 + \frac{\text{n}}{1!} + \frac{\text{n(n+1)}}{2!}x^2 + \frac{\text{n(n+1)(n+2)}}{3!}x^3 + \ldots\ldots -(1+\text{nx})$$

$$= (1-x)^{-n} - (1 + \text{nx})$$

$$= (1 - \frac{6}{8})^{-9/6} - (1 + \frac{9}{6}.\frac{6}{8})$$

$$= (\frac{1}{4})^{-3/2} - (1 + \frac{9}{8})$$

$$= \frac{47}{8}.$$

$$\therefore \quad \text{S} = \frac{47}{9}.$$

Example 4. Find the sum of to infinity of the series $\frac{1}{24} - \frac{1.3}{24.32} + \frac{1.3.5}{24.32.40} - \ldots.$

Solution.

Proceeding as in the previous example, we get

$$\text{S} = \frac{\frac{1}{2}}{3}.\left(\frac{2}{8}\right) + \frac{\frac{1}{2}.\frac{3}{2}}{3.4}.\left(\frac{2}{8}\right)^2 + \frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{3.4.5}.\left(\frac{2}{8}\right)^3 + \ldots..$$

In order to express this in the standard binomial form, the factor 1 . 2 must be inserted in each denominator, and two additional factors must be then inserted in each numerator to secure that the number of factors in the numerator is the same as that in the denominator. In order that the factors of the numerator may remain in A.P. the additional factors(which should be the same in each term) must be $-\frac{3}{2}, \frac{1}{2}$.

$$\therefore \quad -\frac{3}{2}.-\frac{1}{2}.\text{S}.\frac{1}{1.2} = \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}}{1.23}.\left(\frac{2}{8}\right) - \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}.\frac{3}{2}}{1.2.3.4}.\left(\frac{2}{8}\right)^2 + \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{1.2.3.4.5}.\left(\frac{2}{8}\right)^3$$

The index of x should be the same as the number of factors in the numerator.

$\therefore$ The series is to be multiplied by $\left(\frac{2}{8}\right)^2$.

$\therefore\ -\frac{3}{2}.-\frac{1}{2}.S.\frac{1}{2}.\left(\frac{2}{8}\right)^2$

$= \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}}{3!}\left(\frac{2}{8}\right)^3 - \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{4!}.\left(\frac{2}{8}\right)^4 + \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{5!}.\left(\frac{2}{8}\right)^5 + \ldots\ldots$

i.e., $\frac{3S}{128} = \frac{n(n+1)(n+2)}{3!}x^3 - \frac{n(n+1)(n+2)(n+3)}{4!}x^4 + \ldots\ldots$

If $n = -\frac{3}{2}$, $x = \frac{2}{8}$.

$\therefore\ \frac{3S}{128} = -(1+x)^{-n} + \{1 - nx + \frac{n(n+1)}{2!}x^2\}$

$= -(1+\frac{2}{8})^{3/2} + \{1 + \frac{3}{2}.\frac{2}{8} + \frac{-\frac{3}{2}.-\frac{1}{2}}{2!}\left(\frac{2}{8}\right)^2\}$

$= \frac{-5\sqrt{5}}{8} + 1 + \frac{3}{8} + \frac{3}{128}$

$= \frac{179}{128} - \frac{-5\sqrt{5}}{8}$.

$\therefore\ S = \frac{1}{3}(179 - 80\sqrt{5})$.

Exercises

Find the sum to infinity of the following series:

(1) $\frac{3}{1} + \frac{3.5}{1.2}.\frac{1}{3} + \frac{3.5.7}{4.8.12} + \ldots\ldots$

(2) $\frac{3}{50} + \frac{3.18}{50.100} + \frac{3.18.33}{50.100.150} + \ldots\ldots$

(3) $\frac{5}{3.6} + \frac{5.7}{3.6.9} + \frac{5.7.9}{3.6.9.12} + \ldots\ldots$

(4) $\frac{3}{18} + \frac{3.7}{18.24} + \frac{3.7.11}{18.24.30} + \ldots\ldots$

(5) $\frac{5}{3.6}.\frac{1}{4^2} + \frac{5.8}{3.6.9}.\frac{1}{4^3} + \frac{5.8.11}{3.6.9.12}.\frac{1}{4^4} + \ldots.$

(6) $\frac{1}{2^3(3!)} - \frac{1.3}{2^4(4!)} + \frac{1.3.5}{2^5(5!)} + \ldots\ldots$

Answers: 1. $3^{5/3} - 3$, 2. $\left(\frac{10}{7}\right)^{1/5} - 1$, 3. $\sqrt{3} - \frac{2}{3}$, 4.$\frac{1}{5}\left\{8(27)^{1/4} - 17\right\}$,
5. $\frac{1}{2}\left\{\left(\frac{4}{3}\right)^{2/3} - \frac{7}{6}\right\}$, 6.$\frac{23}{24} - \frac{2}{3}\sqrt{2}$.

Sum of coefficients.

If f (x) can be expanded as an ascending series in x, we can find the sum of the list (n+1) coefficients.

Let f (x) = $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots$

$\therefore \frac{f(x)}{1-x} = (a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots) \cdot (1 + x + x^2 + \ldots)$

$\therefore$ Coefficient of x^n in $\frac{f(x)}{1-x} = a_0 + a_1 + a_2 + \ldots + a_n$.

Thus, to find the sum of the first (n+1) coefficients in the expansion of f (x), we have only to find the coefficient of x^n of the expansion of $\frac{f(x)}{1-x}$.

Example 1. Find the sum of the coefficients of the first (r +1) term in the expansion of $(1-x)^{-3}$.

Solution.

The required result is the coefficient of x^r in the expansion of $\frac{(1-x)^{-3}}{1-x}$.

i.e., in the expansion of $(1-x)^{-4}$

i.e., in $1 + 4x + \frac{4.5}{2!}x^2 + \frac{4.5.6}{3!}x^3 + \ldots\ldots + \frac{(r+1)(r+2)(r+3)}{3!}x^r$

$\therefore$ Sum of the (r +1) coefficients in the expansion of

$(1-x)^{-3}$ is $\frac{(r+1)(r+2)(r+3)}{6}$.

Example 2. If *n* is a positive integer and $\frac{(1+x)^n}{(1-x)^3} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots. + a_nx^n + \ldots$

Show that $a_0 + a_1 + a_2 + \ldots + a_{n-1} = \frac{1}{3}$ n (n+2)(n+7) 2^{n-4}.

Solution.

The sum required = coefficient of x^{n-1} in the expansion of $\frac{(1+x)^n}{(1-x)^3(1-x)}$.

= " " $\frac{(1+x)^n}{(1-x)^4}$.

Now $(1+x)^n$ = $\{2-(1-x)\}^n$

$$= 2^n - n \cdot 2^{n-1}(1-x) + \frac{n(n-1)}{2!}2^{n-2}(1-x)^2 - \frac{n(n-1)(n-2)}{3!}2^{n-3}(1-x)^3$$

Involving powers of (1 – x), higher than third.

Hence $\frac{(1+x)^n}{(1-x)^4} = \frac{2^n}{(1-x)^4} - \frac{n \cdot 2^{n-1}}{(1-x)^3} + \frac{n.(n-1)2^{n-2}}{2!(1-x)^2} - \frac{n(n-1)(n-2)}{3!(1-x)}2^{n-3}$

+ an integral expression of (n – 4)th degree.

Coefficient of x^{n-1} in $(1-x)^{-4}$ is $\frac{n(n+1)(n+2)}{3!}$

" $(1-x)^{-3}$ is $\frac{n(n+1)}{2!}$

" $(1-x)^{-2}$ is n.

" $(1-x)^{-1}$ is 1.

Hence the coefficient of x^{n-1} in $\frac{(1+x)^n}{(1-x)^4}$ is

$$\frac{2^n n(n+1)(n+2)}{3!} - \frac{2^{n-1}n^2(n+1)}{2!} + \frac{2^{n-2}n(n-1)}{2!} . n - 2^{n-3}\frac{n(n-1)(n-2)}{3!}$$

$= \frac{2^{n-1}n(n+1)(n+2)}{3} - 2^{n-2}n^2(n+1) + 2^{n-3}n^2(n-1) - 2^{n-4}\frac{n(n-1)(n-2)}{3}$

$= \frac{2^{n-4}}{3}$ n { 8 (n+1) (n+2) – 12n (n+1) + 6n(n– 1) – (n – 1)(n – 2)}

$= \frac{2^{n-4}n}{3}$ { $8n^2$+24n+16 – $12n^2$– 12n+$6n^2$– 6n – n^2 + 3n –2}

$= \frac{2^{n-4}n}{3}$ (n^2 + 9n + 14)

$= \frac{2^{n-4}n(n+2)(n+7)}{3}$

$= \frac{1}{3}n(n+2)(n+7)\frac{2^{n-4}n}{3}$.

Exercises

1. Find the sum of n terms of the series 1+ n + $\frac{n(n+1)}{1.2}$+$\frac{n(n+1)(n+2)}{1.2.3}$+…..
2. Find the sum of the first n+1 coefficients in the expansion of $\frac{2x-4}{(1+x)(1-2x)}$ is ascending powers of x.
3. Show that if a_m be the coefficient of x^m in the expansion of $(1+x)n$, then whatever n be $a0 - a1 + a2 + \ldots + (-1)\,m-1am-1 = \frac{(n-1)(n-2)\ldots(n-m+1)}{(m-1)!}(-1)^{m-1}$.

Answers : 1. $\frac{(2n-1)!}{n!(n-1)!}$,2.$\{1+(-1)^n - 2^{n+2}\}$.

Integro-Binomial Series.

When n is a positive integer we know that

$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots C_rx^r + \ldots$ (1)

Changing n into (n – 1) in (1) , we get

$(1+x)^{n-1} = 1 + \frac{(n-1)}{1!}x + \frac{(n-1)(n-2)}{2!}x^2 + \ldots..$

$\therefore \; n(1+x)^{n-1} = n + \frac{n(n-1)}{1!}x + \frac{n(n-1)(n-2)}{2!}x^2 + \ldots..$

$= C_1 + 2.\, C_2.x + 3\, C_{3.}x^2 + \ldots.r.\, C_{r.}x^{r-1} + \ldots\ldots$ (2)

Changing n into (n – 1) in (2) , we get

$(n-1)\,(1+x)^{n-2} = (n-1) + \frac{(n-1)(n-2)}{1!}x + \frac{(n-1)(n-2)(n-3)}{2!}x^2 + \ldots.$

$\therefore (n-1)\,(1+x)^{n-2} = n(n-1) + \frac{n(n-1)(n-2)}{1!}x + \frac{n(n-1)(n-2)(n-3)}{2!}x^2 + \ldots.$

$= 1.2.\, C_2 + 2.3.\, C_3 x + \ldots.r\,(r-1)\,.\, C_r x^{r-2} + \ldots..$ (3)

Similarly

$n\,(n-1)(n-2)\,(1+x)^{n-3} = 1.2.3.\, C_3 + 2.3.4.\, C_4 x + \ldots.$

$+ r\,(r-1)(r-2)\,.\, C_r x^{r-3} + \ldots..$ (4)

and so on.

If n is not a positive integer , the result (1) , (2) , (3) , (4) , ….. are also true provided $|x| < 1$.

In this case C_0 , C_1 , C_2 , …. do not represent ${}_nC_0$, ${}_nC_1$, ${}_nC_2$, … but $\frac{n_1}{1!}$, $\frac{n_2}{2!}$, $\frac{n_3}{3!}$, …..

In a similar way , we can show that

$\frac{(1+x)^{n+1}}{n+1} = \frac{1}{n+1} + C_0 x + \frac{C_1}{2}x^2 + \ldots. + \frac{C_r}{r+1}x^{r+1} + \ldots.$

$\frac{(1+x)^{n+1}}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} + \frac{1}{n+1}x + \frac{C_0}{1.2}x^2 + \ldots + \frac{C_r}{(r+1)(r+2)}x^{r+2} + \ldots.$

and so on.

The series whose general term is f (r) . $C_r \,.\, x^r$ where f (r) is a polynomial in r is called an **integro-Binomial Series**.

To sum up such a series the following method may be adopted.

Express f (r) = $a_0 + a_1 r + a_2 r(r-1) + \ldots.$

By giving values 0 , 1 , 2 , …. r , a_0 , a_1 , a_2 , ….. can be determined.

$$f(r) . C_r x^r = a_0 C_r . x^r + a_1 r . C_r . x^r + a_2 r(r-1) . C_r . x^r + \ldots$$

$$\therefore \sum_0^\infty f(r). C_r . x^r = a_0 \sum_0^\infty C_r . x^r + a_1 \sum_0^\infty r. C_r . x^r$$

$$+ a_2 \sum_0^\infty r(r-1). C_r . x^r + \ldots\ldots$$

$$= a_0(1+x)^n + a_1 x. \sum_0^\infty r. C_r . x^{r-1}$$

$$+ a_2 x^2 \sum_0^\infty r(r-1). C_r . x^{r-2} + \ldots\ldots$$

$$= a_0(1+x)^n + a_1 x. n(1+x)^{n-1} + a_2 x^2 n(n-1). (1+x)^{n-2} + \ldots..$$

$$= a_0(1+x)^n + a_1 n. x(1+x)^{n-1} + a_2 n. (n-1). x^2 (1+x)^{n-2} + \ldots..$$

Example 1. Sum the series $\sum_0^\infty (r+1)^2 \ C_r . x^r$.

Solution.

Let $(r+1)^2 = a_0 + a_1 r + a_2 r(r-1)$.

Put r = 0 $\therefore$ $a_0 = 1$

r = 1 $\therefore$ $a_1 = 3$.

Equating the coefficients of r^2 on both sides, we get $a_2 = 1$.

$\therefore$ $(r+1)^2 = 1 + 3r + r(r-1)$.

$$\therefore \sum_0^\infty (r+1)^2. C_r x^r = \sum_0^\infty C_r . x^r + 3. \sum_0^\infty r. C_r . x^r + \sum_0^\infty r(r-1). C_r \ x^r$$

$$= \sum_0^\infty C_r . x^r + 3x. \sum_0^\infty r. C_r . x^{r-1} + x^2 \sum_0^\infty r(r-1). C_r \ x^{r-2}$$

$$= (1+x)^n + 3x. n(1+x)^{n-1} + x^2 n(n-1) \ (1+x)^{n-2}$$

$$= (1+x)^{n-2} \{ (1+x)^2 + 3nx(1+x) + n(n-1) \ x^2 \}$$

$$= (1+x)^{n-2} \{ (n+1)^2 x^2 + (3n+2)x + 1 \}.$$

Example 2. If $|x| < 1$, prove that $\frac{1+x}{(1-x)^3} = 1^2 + 2^2.x + 3^2.x^2 + 4^2.x^3 + \ldots\ldots$ to ∞ .

Solution.

The n^{th} term of the series is $(n+1)^2.x^n$.

Express $(n+1)^2$ in the form $a_0 + a_1 n + a_2 n(n-1)$.

$\therefore$ $a_0 = 1$, $a_1 = 3$, $a_2 = 1$.

$\therefore$ $(n+1)^2 = 1 + 3n + n(n-1)$.

$$\therefore \sum_0^\infty (n+1)^2 x^n = \sum_0^\infty x^n + 3\sum_0^\infty n.x^n + \sum_0^\infty n(n-1).\ x^n$$

$$= 1 + x + x^2 \ldots\ldots x^n + \cdots \text{ to } \infty + 3\{x + 2x^2 + 3x^3 + \cdots \infty\}$$

$$+ \{1.2x^2 + 2.3x^3 + 3.4x^4 + \cdots \infty\}$$

$$= (1-x)^{-1} + 3x\{1 + 2x + 3x^2 + \cdots to\ \infty\}$$

$$+ x^2\{1.2 + 2.3x + 3.4x^2 + \cdots to\ \infty\}$$

$$= (1-x)^{-1} + 3x(1-x)^{-2} + x^2.2(1-x)^{-3}$$

$$= \frac{1}{1-x} + \frac{3x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}$$

$$= \frac{(1-x)^2 + 3x(1-x) + 2x^2}{(1-x)^3}$$

$$= \frac{1+x}{(1-x)^3}.$$

Example 3. Sum the series $\sum_0^\infty \frac{r+1}{r+2}.\ C_r x^r$

Solution.

We have $\frac{r+1}{r+2} = \frac{(r+1)^2}{(r+1)(r+2)}$

$$= \frac{a_0 + a_1(r+1) + a_2(r+1)(r+2)}{(r+1)(r+2)}$$

$\therefore\ (r+1)^2 = a_0 + a_1(r+1) + a_2(r+1)(r+2)$

$\therefore\ a_0 = 1,\ a_1 = -1,\ a_2 = 1.$

$$\therefore\ \frac{r+1}{r+2} = \frac{1-(r+2)(r+1)(r+2)}{(r+1)(r+2)}$$

$$= \frac{1}{(r+1)(r+2)} - \frac{1}{(r+1)} + 1.$$

$$\therefore\ \sum_0^\infty \frac{r+1}{r+2} C_r.x^r = \sum_0^\infty \frac{1}{(r+1)(r+2)} C_r x^r - \sum_0^\infty \frac{1}{(r+1)} C_r x^r + \sum_0^\infty C_r x^r$$

$$= \frac{1}{x^2}\sum_0^\infty \frac{1}{(r+1)(r+2)} C_r.x^{r+2} - \frac{1}{x}\sum_0^\infty \frac{1}{(r+1)} C_r.x^{r+1} +$$

$$\sum_0^\infty C_r.x^r.$$

We have learned that

$$\sum_0^\infty \frac{C_r.}{r+1} x^{r+1} + \frac{1}{n+1} = \frac{(1+x)^{n+1}}{n+1}$$

$$\sum_0^\infty \frac{C_r.}{(r+1)(r+2)} x^{r+2} + \frac{1}{(n+1)(n+2)} + \frac{x}{n+1} = \frac{(1+x)^{n+2}}{(n+1)(n+2)}.$$

$$\therefore\ \sum_0^\infty \frac{r+1}{r+2} C_r.x^r = \frac{1}{x^2}\left[\frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{x}{n+1} - \frac{1}{(n+1)(n+2)}\right]$$

$$- \frac{1}{x}\left[\frac{(1+x)^{n+1}}{n+1} + \frac{1}{n+1}\right] + (1+x)^n$$

$$= (1+x)^n\left[1 + \frac{(1+x)^2}{(n+1)(n+2)x^2} - \frac{1+x}{(n+1)x}\right] - \frac{1}{(n+1)(n+2)x^2}$$

$$= \frac{(1+x)^n}{(n+1)(n+2)}\left[(n+1)(n+2) + \frac{(1+x)^2}{x^2} - \frac{(n+2)(1+x)}{x}\right] - \frac{1}{(n+1)(n+2)x^2}$$

$$= \frac{(1+x)^n}{(n+1)(n+2)}\left[x^2 + 3n + 2 + \frac{1}{x^2} + \frac{2}{x} + 1 - \frac{(n+2)}{x} - (n+2)\right] - \frac{1}{(n+1)(n+2)x^2}$$

$$= \frac{(1+x)^n}{(n+1)(n+2)}\left[(n+1)^2 - \frac{n}{x} + \frac{1}{x^2}\right] - \frac{1}{(n+1)(n+2)x^2}.$$

Exercises

1. If $(1 + x + x^2)^2 = a_0 + a_1x + a_2x^2 + \ldots$ prove that

   $a_0 - n.\, a_{r-1} + \frac{n(n-1)}{1.2} a_{r-2} - \ldots. + (-1)^2 . \frac{n!}{r!(n-r)!} a_0 = 0$ unless r is a multiple of 3 .

   What is its value in this case?

2. Express $\frac{1}{1-x-6x^2}$ as the sum of two partial fractions and hence show that

   $1 + (n-1)6 + \frac{(n-2)(n-3)}{2!} 6^2 + \frac{(n-3)(n-4)(n-5)}{3!} 6^3 + \ldots =$

   $\frac{1}{5}\{3^{n+1} + (-1)^{n+1} 2^{n+1}\}$.

3. Show that $1 + 2(n-1) + \frac{2^2(n-2)(n-3)}{1.2} + \frac{2^3(n-3)(n-4)(n-5)}{3!} + \ldots =$

   $\frac{1}{3}\{2^{n+1} + (-1)^n\}$.

Approximate values.

The Binomial series can be used to obtain approximate values and limits of expressions as follows.

Example 1. Find correct to six places of decimals the values of $\frac{1}{(9998)^{1/4}}$.

Solution.

$$\left(\frac{1}{(9998)}\right)^{1/4} = \frac{1}{(10000-2)^{1/4}}$$

$$= \frac{1}{(10^4-2)^{1/4}}$$

$$= \frac{1}{10(1-\frac{2}{10^4})^{1/4}}$$

$$= \frac{(1-\frac{2}{10^4})^{-1/4}}{10}$$

$$= \frac{1 + \frac{1}{4} . \frac{2}{10^4} + \frac{\frac{1}{4} . \frac{5}{4}}{2!} . \frac{4}{10^8} + \cdots.}{10}$$

$$= \frac{1}{10} + \frac{1}{2} \cdot \frac{1}{10^5} + \frac{5}{8} \cdot \frac{1}{10^9} + \ldots.$$

$$= 0.1 + \frac{1}{2}(0.00001) + \frac{5}{8}(0.000000001)$$

= 0.1 + 0.000005 + 0.0000000005

= 0.1000050005

$\therefore \frac{1}{(9998)^{1/4}}$ = 0.100005 correct to six places of decimals.

Example 2. Calculate correct to six places of decimals $(1.01)^{1/2} - (0.99)^{1/2}$.

Solution.

Write x = 0.01.

$$\therefore \quad (1.01)^{1/2} = (1+x)^{1/2}$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{3!}x^3 + \ldots.$$

$$(0.99)^{1/2} = (1-x)^{1/2}$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2!}x^2 - \frac{\frac{1}{2}\cdot(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \ldots.$$

$$\therefore (1.01)^{1/2} - (0.99)^{1/2} = 2\left\{\frac{1}{2}x + \frac{\frac{1}{2}\cdot(-\frac{1}{2})\cdot(-\frac{3}{2})}{3!}x^3 + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{3!}x^5 + \cdots\right\}$$

$$= 2\left\{\frac{1}{2}x + \frac{1}{16}x^3 + \frac{7}{256}x^5 + \cdots\right\}$$

$$= x + \frac{1}{8}x^3 + \frac{7}{128}x^5 + \cdots$$

$$= (0.01) + \frac{1}{8}(0.01)^3 + \frac{7}{128}(0.01)^5 + \cdots$$

$= 0.01 + \frac{1}{8}(0.000001)$ + terms not affecting the 8^{th} decimal place

= 0.01 + 0.000000125

$= 0.010000125$

$\therefore (1.01)^{1/2} - (0.99)^{1/2} = 0.010000$ correct to six places of decimals.

Exercises

1. Find the value of $\frac{1}{(128)^{1/3}}$ correct to five places of decimals.
2. Find the expansion of $(1+\frac{1}{64})^{1/3}$ and find the cube root of 65 correct to three places of decimals.
3. Prove that $(2)^{1/3} = 1\frac{1}{4}(1+0.024)^{1/3}$ and hence find the cube root of two to four places of decimals.
4. Evaluate $\left(\frac{0.998}{1.002}\right)^{1/3}$ correct to four places of decimals, without using logarithms.
5. Find to five places of decimals the value of $(1003)^{\frac{1}{3}} - (997)^{\frac{1}{3}}$.

   Answers: 1. 0.19842, 2. 4.021, 4. 1.0027, 5. 0.02000.

Example 1. When x is small, prove that $\frac{(1-3x)^{-2/3}+(1-4x)^{-3/4}}{(1-3x)^{-1/3}+(1-4x)^{-1/4}} = 1+\frac{3}{2}x + 4x^2$ approximately.

Solution.

The expression is equal to

$$= \frac{1+\frac{2}{3}.3x+\frac{\frac{2}{3}\cdot\frac{5}{3}}{2!}(3x)^2+\frac{\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}}{3!}(3x)^3+\cdots+1+\frac{3}{4}.4x+\frac{\frac{3}{4}\cdot\frac{7}{4}}{2!}(4x)^2+\cdots}{1+\frac{1}{3}.3x+\frac{\frac{1}{3}\cdot\frac{4}{3}}{2!}(3x)^2+\frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}}{3!}(3x)^3+\cdots+1+\frac{1}{4}.4x+\frac{\frac{1}{4}\cdot\frac{5}{4}}{2!}(4x)^2+\cdots}$$

Since x^3 and higher powers of x may be neglected the expression

$$= \frac{2+5x+15\frac{1}{2}x^2}{2+2x+4\frac{1}{2}x^2}$$

$$= \frac{(2+5x+15\frac{1}{2}x^2)}{2(1+x+\frac{9}{4}x^2)}$$

$$= \frac{2+5x+15\frac{1}{2}x^2}{2} \; . \; (1+x+\frac{9}{4}x^2)^{-1}$$

$$= (1+\frac{5}{2}x+\frac{31}{4}x^2)\{ 1+x(1+\frac{9}{4}x)\}^{-1}$$

$$= \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\{1-x\left(1+\frac{9}{4}x\right)+x^2(1+\frac{9}{4}x)^2\ldots\}$$

$$= \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\left(1-x-\frac{9}{4}x^2+x^2\right)$$

(x^3 and higher powers of x neglected)

$$= \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\left(1-x-\frac{5}{4}x^2\right)$$

$$= 1+\frac{5}{2}x+\frac{31}{4}x^2-x-\frac{5}{2}x^2-\frac{5}{4}x^2$$

$$= 1+\frac{3}{2}x+4x^2.$$

Example 2. Show that $\sqrt{x^2+16}-\sqrt{x^2+9} = \frac{7}{2x}$ nearly for sufficiently large values of x.

Solution.

The expression $= (x^2+16)^{1/2}-(x^2+9)^{1/2}$

$$= x(1+\frac{16}{x^2})^{1/2}-x(1+\frac{9}{x^2})^{1/2}$$

$$= x(1+\frac{1}{2}.\frac{16}{x^2}-\cdots)-x(1+\frac{1}{2}.\frac{9}{x^2}-\cdots)$$

(Since $\frac{1}{x}$ is small, the expansion is valid)

$$= x+\frac{8}{x}-x-\frac{9}{2x}$$

$= \frac{7}{2x}$ nearly.

Exercises

1. If x be so small that its square and higher powers may be neglected, find the value of

   $(1-7x)^{1/3}-(1+2x)^{-3/4}$

2. When x is small , show that $\frac{(1-x)^{-5/2}+(16+8x)^{1/2}}{(1+x)^{-1/2}+(2+x)}= 1+\frac{23}{40}x^2$ approximately.
3. If x be so small that its squares and higher powers may be neglected. Prove that $\frac{(9+2x)^{1/2}+(3+4x)}{(1-x)^{1/3}}=9+\frac{74}{5}x$ nearly.
4. If x be so small that powers of x above x^3 may be neglected, show that $\frac{(1+x+x^2)+(1+x)^2}{(1-x)^{1/3}}= 1+4x+7x^2+6x^3.$
5. If c is small in comparison with *l*, show that $(\frac{l}{l+c})^{1/2}+(\frac{l}{l-c})^{1/2}=2+\frac{3c^3}{4l^2}$

   approximately.
6. Show that $\sqrt{x^2+4}-\sqrt{x^2+1}$ is $1-\frac{1}{4}x^2+\frac{7}{64}x^4$ nearly when x is small and $\frac{3}{2x}(1-\frac{3}{4x^2}+\frac{3}{8x^4})$ nearly when x is large.

   Answer: $1.1-\frac{23}{6}x.$

Extra problem

1. Find the general term in the expansion $(4-7x)^{-2/5}$ starting when will the expansion be valid.

   Solution.

   $(4-7x)^{-2/5} = 4^{-2/5}\left(1-\frac{7x}{4}\right)^{-2/5} = 2^{-4/5}\left(1-\frac{7x}{4}\right)^{-2/5}$

   $\left(1-\frac{7x}{4}\right)^{-2/5}$ can be expanded in binomial series if $\left|\frac{7x}{4}\right| < 1$. (i.e) if $|x|<\frac{4}{7}$

   The general term T_{r+1} in $\left(1-\frac{7x}{4}\right)^{-2/5}$ is

   $=\frac{-\frac{2}{5}\left(-\frac{2}{5}-1\right)\left(-\frac{2}{5}-2\right).....\left(-\frac{2}{5}-r+1\right)\left(-\frac{7x}{4}\right)^r}{r!}$

   $=\frac{(-2)(-7)(-12).....(-5r+3)}{5^r r!}(-1)^r\left(\frac{7x}{4}\right)^r$

   $=\frac{2.7.12.....(5r-3)}{r!}(-1)^{2r}\left(\frac{7x}{20}\right)^r$

   $=\frac{2.7.12.....(5r-3)}{r!}\left(\frac{7x}{20}\right)^r$

The general term in $(4 - 7x)^{-2/5}$ is $(2)^{-4/5} \frac{2.7.12......(5r-3)}{r!} \left(\frac{7x}{20}\right)^r$.

2. If $|x| < \frac{1}{2}$ prove that coefficient of x^n in the expansion of $(2 - 4x)(1 - 2x)^{-2}$ is 2^{n+1}

Solution.

$|x| < \frac{1}{2} \Rightarrow |2x| < 1$

Hence we can expand (1 – 2x)-2 in binomial series.

Now, $(2 - 4x)(1 - 2x)^{-2} = (2 - 4x) \sum_0^\infty (r+1)(2x)^r$

$= (2 - 4x)[1 + 2(2x) + 3(2x)^2 + ... + n\,(2x)^{n-1} + (n+1)(2x)^n + ...]$

$= (2 - 4x)[1 + 2.2x + 3.2^2x^2 + ... + n\,2^{n-1}x^{n-1} + (n+1)2^nx^n + ...]$

Coefficient of $x^n = 2(n+1)2^n - 4\,(n.2^{n-1})$

$= n.2^{n+1} + 2^{n+1} - n\,2^{n+1} - n\,2^{n+1} = 2^{n+1}$.

3. Find the coefficient of x^n in the expansion $(1 - 2x + 3x^2 - 4x^3 + ...)^{-n}$

Solution.

$(1 - 2x + 3x^2 - 4x^3 + ...)^{-n} = [(1+x)^{-2}]^{-n}$

$= (1+x)^{2n}$

Coefficient of x^n in $(1 - 2x + 3x^2 - 4x^3 + ...)^{-n}$ is same as the coefficient of x^n in $(1+x)^{2n}$

and it is

$= \frac{2n(2n-1)(2n-2).....(2n-\overline{n-1})}{n!}$

$= \frac{2n(2n-1)(2n-2)....(n+1)}{n!}$

$= \frac{2n(2n-1)......(n+1)[n(n-1).......2.1]}{n![1.2......(n-1)n]}$

$= \frac{2n!}{n!n!}$.

4. Find the coefficient of x^n when $\frac{7+x}{(1+x)(1+x^2)}$ is expanded in ascending power of x.

Solution.

Let $\frac{7+x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+c}{1+x^2}$.

We can find A = 3; B = – 3; C = 4.

Therefore $\frac{7+x}{(1+x)(1+x^2)} = \frac{3}{1+x} + \frac{4-3x}{1+x^2}$.

$= 3(1+x)^{-1} + (4 - 3x)(1 + x^2)^{-1}$

$= 3(1 - x + x^2 - x^3 + ...) + (4 - 3x)(1 - x^2 + x^4 - x^6 + ...)$

Case 1. r is an odd integer say r = 2n + 1, n∈N.

Therefore coefficient of x^r = coeff.of x^{2n+1}

$= -3 + (-3)(-1)^n$

$= -3 + (-3)(-1)^{(r-1)/2}$

Case 2.r is an even integer say r = 2n, n∈N.

Therefore coefficient of x^r = coeff.of x^{2n}

= $-3 + 4(-1)^n$

= $-3 + 4(-1)^{r/2}$.

5. If x is so small that its square and higher powers may be neglected prove that

$\frac{\sqrt{1+x}(4-3x)^{3/2}}{(8+5x)^{1/3}} = 4 - \frac{10x}{3}$.(nearly)

Solution.

$\frac{\sqrt{1+x}(4-3x)^{3/2}}{(8+5x)^{1/3}} = (1+x)^{1/2}4^{3/2}\left(1-\frac{3x}{4}\right)^{3/2}8^{1/3}\left(1+\frac{5x}{8}\right)^{-1/3}$

$= 4\left(1+\frac{1}{2}x+\cdots\right)\left(1-\frac{9}{8}x+\cdots\right)\left(1-\frac{5x}{24}+\cdots\right)$

$= 4\left[1+x\left(\frac{1}{2}-\frac{9}{8}-\frac{5}{24}\right)\right]$ (neglecting x^2 and higher power of x)

$= 4-\frac{10x}{3}$.

6. Show that $1 + n\left(\frac{2a}{1+a}\right) + \frac{n(n+1)}{1.2}\left(\frac{2a}{1+a}\right)^2 + \ldots\ldots = \left(\frac{1+a}{1-a}\right)^n$.

Solution.

Put $\frac{2a}{1+a} = y$.

Then L.H.S = $1+\frac{ny}{1!}+\frac{n(n+1)}{2!}y^2+\cdots$

$= (1-x)^{-p/q}$ where p = n; a = 1 and $\frac{x}{a} = y$. Hence x = y.

Hence L.H.S = $(1-y)^{-n} = \left(1-\frac{2a}{1+a}\right)^{-n} = \left(\frac{1-a}{1+a}\right)^{-n} = \left(\frac{1+a}{1-a}\right)^n$ = R.H.S

7. Prove that $1+\frac{2n}{3}+\frac{2n(2n+2)}{3.6}+\frac{2n(2n+2)(2n+4)}{3.6.9}+\cdots = 2\left[1+\frac{n}{3}+\frac{n(n+1)}{3.6}+\frac{n(n+1)(n+2)}{3.6.9}+\right.$ $\ldots$

Solution.

L.H.S = $1+\frac{n}{1!}\left(\frac{2}{3}\right)+\frac{n(n+1)}{2!}\left(\frac{2}{3}\right)^2+\ \ldots$

$= \left(1-\frac{2}{3}\right)^{-n} = \left(\frac{1}{3}\right)^{-n} = 3^n$

R.H.S = $2^n\left[1+\frac{n}{1!}\left(\frac{1}{3}\right)+\frac{n(n+1)}{2!}\left(\frac{1}{3}\right)^2+\ \ldots\right]$

$= 2^n\left(1-\frac{1}{3}\right)^{-n} = 2^n\left(\frac{2}{3}\right)^{-n} = 3^n$

L.H.S = R.H.S.

9. Sum to infinity the series $1 + \frac{1}{5} + \frac{1.4}{5.10} + \frac{1.4.7}{5.10.15} + ...$

Solution.

Let S = $1 + \frac{1}{5} + \frac{1.4}{5.10} + \frac{1.4.7}{5.10.15} + ...$

Therefore S = $1 + \frac{1}{1!}\left(\frac{1}{5}\right) + \frac{1.4}{2!}\left(\frac{1}{5}\right)^2 + \frac{1.4.7}{3!}\left(\frac{1}{5}\right)^3 + \cdots$

$= (1-x)^{-p/q}$ where p = 1; q = 3 and $\frac{x}{q} = \frac{1}{5}$. Hence $x = \frac{3}{5}$

Therefore S = $\left(1 - \frac{3}{5}\right)^{-1/3} = \left(\frac{2}{5}\right)^{-1/3} = \left(\frac{5}{2}\right)^{1/3}$.

10. Sum to ∞ the series $\left(\frac{1}{2}\right)^2 + \frac{1}{2!}\left(\frac{1}{2}\right)^4 + \frac{1.3}{3!}\left(\frac{1}{2}\right)^6 + \cdots$

Solution.

Let S = $\left(\frac{1}{2}\right)^2 + \frac{1}{2!}\left(\frac{1}{2}\right)^4 + \frac{1.3}{3!}\left(\frac{1}{2}\right)^6 + \cdots$

Therefore S = $\frac{1}{1!}\left(\frac{1}{4}\right) + \frac{1}{2!}\left(\frac{1}{4}\right)^2 + \frac{1.3}{3!}\left(\frac{1}{4}\right)^3 + \cdots$

-S = $\frac{-1}{1!}\left(\frac{1}{4}\right) + \frac{-1.1}{2!}\left(\frac{1}{4}\right)^2 + \frac{-1.1.3}{3!}\left(\frac{1}{4}\right)^3 + \cdots$

-S + 1 = $1 + \frac{-1}{1!}\left(\frac{1}{4}\right) + \frac{-1.1}{2!}\left(\frac{1}{4}\right)^2 + \frac{-1.1.3}{3!}\left(\frac{1}{4}\right)^3 + \cdots$

$= (1-x)^{-p/q}$ where p = 1; q = 2 and $\frac{x}{q} = \frac{1}{4}$

Hence $x = \frac{1}{2}$. Hence –S+1 = $\left(1 - \frac{1}{2}\right)^{1/2}$

$= \left(\frac{1}{2}\right)^{1/2} = \frac{1}{\sqrt{2}}$. Hence S = $1 - \frac{1}{\sqrt{2}}$.

11. Sum to ∞ the series $\frac{3}{18} + \frac{3.7}{18.24} + \frac{3.7.11}{18.24.30} + ...$

Solution.

Let S = $\frac{3}{18} + \frac{3.7}{18.24} + \frac{3.7.11}{18.24.30} + ...$

$= \frac{3}{3}\left(\frac{1}{6}\right) + \frac{3.7}{3.4}\left(\frac{1}{6}\right)^2 + \frac{3.7.11}{3.4.5}\left(\frac{1}{6}\right)^3 + \cdots$

Therefore $\frac{S(-5)(-1)}{1.2}\left(\frac{1}{6}\right)^2 = \frac{(-5)(-1)3}{3!}\left(\frac{1}{6}\right)^2 + \frac{(-5)(-1)3.7}{4!}\left(\frac{1}{6}\right)^4 + \cdots$

$$\frac{5S}{72} + 1\frac{(-5)}{1!}\left(\frac{1}{6}\right) + \frac{(-5)(-1)}{2!}\left(\frac{1}{6}\right)^2$$

$$= 1 + \frac{(-5)}{1!}\left(\frac{1}{6}\right) + \frac{(-5)(-1)}{2!}\left(\frac{1}{6}\right)^2 + \frac{(-5)(-1)3}{3!}\left(\frac{1}{6}\right)^3 + \cdots$$

$\frac{5S}{72} + \left(1 - \frac{5}{6} + \frac{5}{72}\right) = (1 - x)^{-p/q}$ where p = 5;q = 4 and $\frac{x}{q} = \frac{1}{6}$ and hence x = $\frac{2}{3}$

Therefore $\frac{5S}{72} + \frac{17}{72} = \left(1 - \frac{2}{3}\right)^{5/4}$

$$\therefore \frac{5S}{72} = \left(\frac{1}{3}\right)^{5/4} - \frac{17}{72}$$

$$\therefore S = \frac{72}{5}\left[\frac{3^{-5/4}(72) - 17}{72}\right] = \frac{72}{5}\left[\frac{3^{-5/4}(3)^2 8 - 17}{72}\right]$$

$$= \frac{72}{5}\left[\frac{3^{3/4}(8) - 17}{72}\right] = \frac{72}{5}\left[\frac{8(27)^{1/4} - 17}{72}\right]$$

$$S = \frac{1}{5}\left(8(27)^{1/4} - 17\right).$$

Exponential Series

We will learn some series which can be summed up by exponential series. We have proved that for all real values of x.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + .. \text{ to } \infty \qquad \ldots\ldots\ldots\ldots..(1)$$

In particular when x = 1 , we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + .. \text{ to } \infty \qquad \ldots\ldots\ldots\ldots..(2)$$

and when x = −1 , we have

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n . \frac{1}{n!} + .. \text{ to } \infty \qquad \ldots\ldots\ldots\ldots..(3)$$

Changing x into $-x$ in series (1) , we get

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots + (-1)^n . \frac{x^n}{n!} + \cdots \qquad \ldots\ldots\ldots\ldots..(4)$$

Adding (1) and (4) , we get

$$\frac{e^x - e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \text{ to } \infty \qquad \ldots\ldots\ldots\ldots\ldots(5)$$

Subtracting (4) from (1) , we get

$$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \text{ to } \infty \qquad \ldots\ldots\ldots\ldots(6)$$

When x = 1 , series (5) and (6) become

$$\frac{e+e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \cdots \text{ to } \infty \qquad \ldots\ldots\ldots\ldots(7)$$

$$\frac{e-e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots \text{ to } \infty \qquad \ldots\ldots\ldots\ldots(8)$$

Note. It can be verified that e is an irrational number whose value lies between 2 and 3. Further the value of e correct to four places of decimals is given by e = 2.7183. We shall use these series to find the sums of certain series. The different methods are illustrated by the following worked examples..

Example. Sum the series $1 + \frac{1+3}{2!} + \frac{1+3+3^3}{3!} + \frac{1+3+3^2+3^3}{4!} + \cdots$ to ∞.

Solution.

Let u_n be the n^{th} term of the series and S be the sum to infinity of the series.

$$\therefore \quad u_n = \frac{1+3+3^2+\ldots\ldots\ldots+3^{n-1}}{n!}$$

$$= \frac{3^n-1}{3-1} \cdot \frac{1}{n!}$$

$$= \frac{1}{2}\left(\frac{3^n}{n!} - \frac{1}{n!}\right)$$

$$\therefore \quad u_1 = \frac{1}{2}\left(\frac{3^1}{1!} - \frac{1}{1!}\right)$$

$$u_2 = \frac{1}{2}\left(\frac{3^2}{2!} - \frac{1}{2!}\right)$$

$$u_3 = \frac{1}{2}\left(\frac{3^3}{3!} - \frac{1}{3!}\right)$$

$\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots$

$$u_n = \frac{1}{2}\left(\frac{3^n}{n!} - \frac{1}{n!}\right)$$

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$$S = \frac{1}{2}\left(\frac{3^1}{1!} + \frac{3^2}{2!} + \cdots + \frac{3^n}{n!} + \cdots\right) - \frac{1}{2}\left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots\right)$$

$$= \frac{1}{2}\,(e^3 - 1) - \frac{1}{2}(e - 1)$$

$$= \frac{1}{2}e(e^2 - 1).$$

Exercises

1. Show that $(1 + \frac{1}{2!} + \frac{1}{4!} + \cdots)^2 = (1 + \frac{1}{3!} + \frac{1}{5!} + \cdots)^2$
2. Show that $\frac{e+1}{e-1} = \frac{\frac{1}{1!}+\frac{1}{3!}+\cdots}{\frac{1}{2!}+\frac{1}{4!}+\cdots}$.
3. Show that $2\,\{1 + \frac{(\log_e n)^2}{2!} + \frac{(\log_e n)^4}{4!} + \ldots\} = \left(n + \frac{1}{n!}\right)$.
4. Show that $\sum_1^\infty \frac{n-1}{n!} = 1$.

If the given series is $\sum_{n=0}^{\infty} f(n).\frac{x^n}{n!}$ where f (n) is a polynomial in n of degree r , we can find constants $a_0, a_1, \ldots\ldots a_r$ so that

$f(n) = a_0 + a_1 n + a_2 n(n-1) \ldots. + a_r n(n-1) \ldots.(n-r+1)$ and then

$$\sum_{n=0}^{\infty} f(n).\frac{x^n}{n!} = a_0 \sum_{n=0}^{\infty}\frac{x^n}{n!} + a_1 \sum_{n=0}^{\infty}\frac{x^n}{(n-1)!} + \ldots. + a_r \sum_{n=0}^{\infty}\frac{x^n}{(n-r)!}$$

$$= a_0.e^x + a_1 x.e^x + \ldots.. + a_r.x^r.e^x$$

$$= (a_0 + a_1 x + a_2 x^2 + \ldots.a_r x^r)\,e^x$$

Example 1. Sum the series $\sum_{n=0}^{\infty}\frac{(n+1)^3}{n!}.\ x^n$.

Solution.

Put $(n+1)^3 = A + Bn + Cn(n-1) + Dn(n-1)(n-2)$.

Putting $n = 0, 1, 2$ and equating the coefficients of n^3, we get

A = 1, B = 7, C = 6, D = 1.

Let the sum of the series be S.

$$S = \sum_0^\infty \frac{1+7n+6n(n-1)+n(n-1)(n-2)}{n!} x^n$$

$$= \sum_0^\infty \frac{x^n}{n!} + 7\sum_0^\infty \frac{x^n}{(n-1)!} + 6\sum_0^\infty \frac{x^n}{(n-2)!} + \sum_0^\infty \frac{x^n}{(n-3)!}$$

Now $\sum_0^\infty \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x$

$$\sum_0^\infty \frac{x^n}{(n-1)!} = x + \frac{x^2}{1!} + \frac{x^3}{2!} \ldots = x.e^x$$

$$\sum_0^\infty \frac{x^n}{(n-2)!} = x^2 + \frac{x^3}{1!} + \frac{x^4}{2!} \ldots = x^2.e^x$$

$$\sum_0^\infty \frac{x^n}{(n-3)!} = x^3 + \frac{x^4}{1!} + \frac{x^5}{2!} \ldots = x^3.e^x$$

∴ $S = (1 + 7x + 6x^2 + x^3)\, e^x$.

Example 2. Sum the series $\frac{1^2}{1!} + \frac{1^2+2^2}{2!} + \frac{1^2+2^2+3^2}{3!}$.........+ $\frac{1^2+2^2+\cdots+n^2}{n!}$+.....

Solution.

Let the n^{th} term of the series be u_n and the sum to infinity be S.

Then $u_n = \frac{1^2+2^2+\cdots+n^2}{n!} = \frac{n(n+1)(2n+1)}{6}\frac{1}{n!}$

Let $n(n+1)(2n+1) = A + Bn + Cn(n-1) + Dn(n-1)(n-2)$.

∴ A = 0, B = 6, C = 9, D = –2.

$$\therefore\ S = \sum_{n=1}^\infty \frac{6n+9n(n-1)+2n(n-1)(n-2)}{6}\frac{1}{n!}$$

$$= \sum_{n=1}^\infty \frac{1}{(n-1)!} + \frac{3}{2}\sum_{n=1}^\infty \frac{1}{(n-2)!} + \frac{1}{3}\sum_{n=1}^\infty \frac{1}{(n-3)!}$$

$= e + \frac{3}{2}e + \frac{1}{3}e$

$= \frac{17}{6}e.$

Exercises

1. Show that the sum to infinity of the series

   $2^2 + \frac{3^2}{1!}x + \frac{4^2}{2!}x^2 + \frac{5^2}{3!}x^3 + \ldots = e^x(x^2 + 5x + 4).$

2. Find the sum to infinity of the series

   (1) $\frac{3.5}{1!}x + \frac{4.6}{2!}x^2 + \frac{5.7}{3!}x^3 + \ldots\infty$

   (2) $1.2 + 2.3x + 3.4.\frac{x^2}{2!} + 4.5.\frac{x^3}{3!}\ldots$

3. Sum to infinity the following series:-

   (1) $1 + \frac{1+2}{2!} + \frac{1+2+3}{3!} + \frac{1+2+3+4}{4!} + \ldots.$

   (2) $\frac{1^4}{1!} + \frac{2^4}{2!} + \frac{3^4}{3!} + \ldots$

   (3) $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \ldots.$

   (4) $\frac{1.2}{1!} + \frac{2.3}{2!} + \frac{3.4}{3!} + \frac{4.5}{4!} + \ldots.$

4. Show that

   (1) $5 + \frac{2.6}{1!} + \frac{3.7}{2!} + \frac{4.8}{3!} + \ldots.$ to $\infty = 13e.$

   (2) $\frac{1^2.\ 2^2}{1!} + \frac{2^2.\ 3^2}{2!} + \frac{3^2.\ 4^2}{3!} + \ldots$ to $\infty = 27e.$

   (3) $\sum_{n=1}^{\infty} \frac{n^3-n+1}{n!} = 5e - 1.$

   Answers : 2.(1).$(x^2 + 7x + 8)\ e^x$, (2). $(x^2 + 4x + 2)e^x$, 3.(1).$\frac{3e}{2}$,(2).15e, (3).e + 1,(4). 3e.

Example 1. Sum the series $\sum_{n=1}^{\infty} \frac{n^2+3}{n+2} \cdot \frac{x^n}{n!}$.

Solution.

Let the sum of the series be S.

Then S = $\sum_{n=1}^{\infty} \frac{(n^2+3)(n+1)}{(n+2)!} \cdot x^n.$

Let $(n^2+3)(n+1) = \mathrm{A} + \mathrm{B}(n+2) + \mathrm{C}(n+2)(n+1) + \mathrm{D}(n+2)(n+1)n.$

We can easily find that A = -7, B = 7, C = -2 and D = 1.

Then S = $\sum_{n=1}^{\infty} \frac{-7+7(n+2)-2(n+2)(n+1)+(n+2)(n+1)n}{(n+2)!} \cdot x^n.$

$$= -7\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!} + 7.\sum_{n=1}^{\infty} \frac{x^n}{(n+1)!} \; -2\sum_{n=1}^{\infty} \frac{x^n}{n!} + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

Now $\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!} = \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^n}{(n+2)!} + \ldots$

$$= \frac{1}{x^2}\left(e^x - 1 - x - \frac{x^2}{2!}\right).$$

$\sum_{n=1}^{\infty} \frac{x^n}{(n+1)!} = \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots + \frac{x^n}{(n+1)!} + \ldots$

$$= \frac{1}{x}(e^x - 1 - x).$$

$\sum_{n=1}^{\infty} \frac{x^n}{n!} = \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = e^x - 1.$

$\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = x + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{(n-1)!} + \ldots = xe^x$

$\therefore$ S = $\frac{-7}{x^2}\left(e^x - 1 - x - \frac{x^2}{2!}\right) + \frac{7}{x}(e^x - 1 - x) - 2\,(e^x - 1) + xe^x$

$$= \frac{e^x}{x^2}(x^3 - 2x^2 + 7x - 7) + \frac{7}{2x^2}(3x^2 + 2).$$

Example 2. Sum the series $\frac{5}{1!} + \frac{7}{3!} + \frac{9}{5!} + \ldots.$

Solution.

The n^{th} term $u_n = \frac{(2n+3)}{(2n-1)!}$

Put $2n + 3 = \mathrm{A}(2n-1) + \mathrm{B}$.

Then A = 1 and B = 4.

$\therefore\ u_n = \frac{2n-1+4}{(2n-1)!}$

$= \frac{2n-1}{(2n-1)!} + \frac{4}{(2n-1)!}$

$= \frac{1}{(2n-2)!} + \frac{4}{(2n-1)!}$

$\therefore\ u_1 = 1 + \frac{4}{1!}$

$u_2 = \frac{1}{2!} + \frac{4}{3!}$

$u_3 = \frac{1}{4!} + \frac{4}{5!}$

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Sum to infinity = $\left(1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\right) + 4\left(\frac{1}{1!} + \frac{1}{3!} + \ldots\right)$

$= \frac{1}{2}\left(e + \frac{1}{e}\right) + 4 \cdot \frac{1}{2} \cdot \left(e - \frac{1}{e}\right)$

$= \frac{5}{2}e - \frac{3}{2e}.$

Example 3. Prove that the infinite series $\frac{2\frac{1}{2}}{1!} - \frac{3\frac{1}{3}}{2!} + \frac{4\frac{1}{4}}{3!} - \frac{5\frac{1}{5}}{4!} + \ldots.. = \frac{1+e}{e}.$

Solution.

Let u_n be the n^{th} term of the series and S be the sum of the series to infinity.

Then $u_n = (-1)^{n+1} \frac{(n+1)\frac{1}{n+1}}{n!}$

$= (-1)^{n+1} \frac{(n+1)^2+1}{(n+1)!}.$

Put $n^2 + 2n + 2$ = A + B (n+1) + C (n+1) n.

$\therefore$ A = 1 , B = 1 , C = 1.

$\therefore$ $u_n = (-1)^{n+1} \frac{1+(n+1)+(n+1)n}{(n+1)!}$

$= (-1)^{n+1} . \left\{\frac{1}{(n+1)!} + \frac{1}{n!} + \frac{1}{(n-1)!}\right\}.$

$\therefore$ $S = \sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{(n+1)!} + \sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{n!} + \sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{(n-1)!}$

Now $\sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{(n+1)!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \ldots. = e^{-1}$

$\sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{n!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} \ldots = -e^{-1} + 1$

$\sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{(n-1)!} = 1 - \frac{1}{1!} + \frac{1}{2!} \ldots = e^{-1}.$

$\therefore$ $S = 1 + e^{-1}$

$= \frac{e+1}{e}.$

Exercises

1. Show that

   (1) $\sum_{n=1}^{\infty} \frac{n-1}{n+2} . \frac{x^n}{n!} = \frac{1}{x^2}\{(x^2 - 3x - 3)\, e^x + \frac{1}{2}x^2 - 3\}.$

   (2) $\sum_{n=1}^{\infty} \frac{(2n-1)}{(n+3)n!} = \frac{1}{2}(43 - 15e)$

2. Sum to infinity the series

   (1) $\frac{3}{1!} + \frac{4}{3!} + \frac{5}{5!} + \frac{6}{7!} + \ldots..$

   (2) $\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \ldots..$

   (3) $\frac{3}{2!} + \frac{5}{4!} + \frac{7}{6!} + \frac{9}{8!} + \ldots..$

3. Show that $\sum_{0}^{\infty} \frac{5n+1}{(2n+1)!} = \frac{e}{2} + \frac{2}{e}$

4. Prove that $\frac{2^2}{1!} + \frac{2^4}{3!} + \frac{2^6}{5!} = \frac{e^4-1}{e^2}.$

5. Show that $\log_e 2 - \frac{1}{2!}(\log_e 2)^2 + \frac{1}{3!}(\log_e 2)^3 \ldots\ldots = \frac{1}{2}$.

Answer : 2(1).$\frac{1}{e}$,(2).$\frac{1}{2}(3e - 2e^{-1})$,(3).$\frac{1}{2e}$.

By equating the coefficients of like powers of x in the expansions of function of x in two different ways, we can derive some identities. The following examples will illustrated the method:

Example 1. By expanding $(e^x - 1)^n$ in two ways or otherwise prove that

$n^r - {}_nC_1(n-1)^r + {}_nC_2(n-2)^r - \ldots\ldots = 0$ where r < n.

What is the sum of the above series when r = n?

Solution.

$$(e^x - 1)^n = e^{nx} - {}_nC_1 e^{(n-1)x} + \ldots.$$

$$= 1 + nx + \frac{(nx)^2}{1!} + \ldots \frac{(nr)^r}{r!} + \ldots - {}_nC_1\left[1 + (n-1)x + \frac{\{(n-1)x\}^2}{2!} + \cdots \frac{\{(n-1)x\}^r}{r!} + \ldots\right.$$

$$\left. + {}_nC_2\left[1 + (n-2)x + \frac{\{(n-2)x\}^2}{2!} + \cdots \frac{\{(n-2)x\}^r}{r!} + \cdots\right]\ldots.\right.$$

Coefficient of x^r in the expansion of $(e^x - 1)^n$

$$= \frac{n^r}{r!} - {}_nC_1 . \frac{(n-1)^r}{r!} + {}_nC_2 . \frac{(n-2)^r}{r!} - \ldots\ldots$$

$$= \frac{1}{r!}\{n^r - {}_nC_1(n-1)^r + {}_nC_2(n-2)^r \ldots.\}$$

Again $(e^x - 1)^n = (1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{x^n}{n!} + \cdots - 1)^n$

$$= \left(\frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{x^n}{n!} + \cdots\right)^n$$

$$= x^n\left(\frac{1}{1!} + \frac{x}{2!} + \cdots \frac{x^{n-1}}{n!} + \cdots\right)^n.$$

All terms in the expansion contain x^n and the higher power of x.

$\therefore$ If r < n , there will be no term containing x^r in the expansion.

$\therefore \frac{1}{r!}\{n^r - {}_nC_1(n-1)^r + {}_nC_2(n-2)^r \ldots\} = 0$

i.e., $n^r - {}_nC_1(n-1)^r + {}_nC_2(n-2)^r \ldots = 0$

If r = n , then

$\frac{1}{n!}\{n^n - {}_nC_1(n-1)^n + {}_nC_2(n-2)^n \ldots\}$

= Coefficient of x^n in the expansion of $x^n\left(\frac{1}{1!} + \frac{x}{2!} + \cdots\right)^n$

= 1 .

$\therefore n^n - {}_nC_1(n-1)^n + {}_nC_2(n-2)^n \ldots = n!$

Example 2. Show that if a^r be the coefficient of x^n in the expansion of e^{e^x} , then

$a_r = \frac{1}{r!}\left\{\frac{1^r}{1!} + \frac{2^r}{2!} + \frac{3^r}{3!}\right\}.$

Hence show that

$(i)\ \frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \ldots = 5e$

$(ii)\ \frac{1^4}{1!} + \frac{2^4}{2!} + \frac{3^4}{3!} + \ldots = 15e.$

Solution.

$e^{e^x} = 1 + e^x + \frac{(e^x)^2}{2!} + \frac{(e^x)^3}{3!} + \frac{(e^x)^4}{4!} + \ldots$

$= 1 + e^x + \frac{e^{2x}}{2!} + \frac{e^{3x}}{3!} + \frac{e^{4x}}{4!} + \ldots$

$= 1 + (1 + x + \frac{x^2}{2!} + \cdots \frac{x^r}{r!} + \ldots) + \frac{1}{2!}(1 + 2x + \frac{2^2x^2}{2!} + \cdots \frac{2^rx^r}{r!} + \ldots)$

$+ \frac{1}{3!}(1 + 3x + \frac{3^2x^2}{2!} + \cdots \frac{3^rx^r}{r!} + \ldots) + \ldots$

Hence the coefficient of $x^r = \frac{1}{r!}\left\{\frac{1^r}{1!}+\frac{2^r}{2!}+\frac{3^r}{3!}\right\}$.

Again

$$e^{e^x} = e^{1+x+\frac{x^2}{2!}+\cdots} = e.e^{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}$$

$$= e.\{1+(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)+\frac{1}{2!}(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)^2$$

$$+\frac{1}{3!}(x+\frac{x^2}{2!}+\cdots)^3+\ldots\}$$

Coefficient of $x^3 = e\left(\frac{1}{3!}+\frac{1}{2!}.2.\frac{1}{2!}+\frac{1}{3!}\right)$

$$= \frac{e}{3!}(1+3+1) = \frac{5e}{3!}.$$

We have shown that the coefficient of x^3

$$= \frac{1}{3!}\left(\frac{1^3}{1!}+\frac{2^3}{2!}+\cdots\right)$$

$$\therefore \quad \frac{1}{3!}\left(\frac{1^3}{1!}+\frac{2^3}{2!}+\frac{3^3}{3!}+\cdots\right) = \frac{5e}{3!}$$

$$\therefore \quad \frac{1^3}{1!}+\frac{2^3}{2!}+\frac{3^3}{3!}+\ldots.. = \frac{5e}{3!}.$$

Similarly equating the coefficient of x^4 , we get the second result.

Example 3. Prove that if n is a positive integer

$$1-\frac{n}{1^2}x+\frac{n(n-1)}{1^2.2^2}x^2-\frac{(n-1)(n-2)}{1^2.2^2.3^2}x^3+\cdots$$

$$= e^x\{1-\frac{n+1}{1^2}x+\frac{(n+1)(n+2)}{1^2.2^2}x^2-\frac{(n+1)(n+2)(n+3)}{1^2.2^2.3^2}x^3+\ldots.\}.$$

Solution.

$$e^y = 1+\frac{y}{1!}+\frac{y^2}{2!}+\frac{y^3}{3!}+\ldots..$$

$(1-\frac{x}{y})^n = 1 - n.\frac{x}{y} + \frac{n(n-1)}{2!}.(\frac{x}{y})^2 - \frac{n(n-1)(n-2)}{3!}(\frac{x}{y})^3 + \ldots..$

$\therefore\ 1 - \frac{n}{1^2}x + \frac{n(n-1)}{1^2.2^2}x^2 - \frac{(n-1)(n-2)}{1^2.2^2.3^2}x^3 + \cdots$

= the term independent of y in the product of $e^y(1-\frac{x}{y})^n$.

$e^y(1-\frac{x}{y})^n = e^x.\,e^{y-x}.\,\frac{(y-x)^n}{y^n}$

$= e^x.\{\,1 + \frac{(y-x)}{1!} + \frac{(y-x)^2}{2!} + \ldots.\}\,\frac{(y-x)^n}{y^n}$

$= e^x\left\{\frac{(y-x)^n + \frac{(y-x)^{n+1}}{1!} + \frac{(y-x)^{n+2}}{2!} + \cdots}{y^n}\right\}$

The term containing y^n in the expression

$(y-x)^n + \frac{(y-x)^{n+1}}{1!} + \frac{(y-x)^{n+2}}{2!} + \cdots$

is $y^n - \frac{{}^{n+1}C_1}{1!}y^n.x + \frac{{}^{n+2}C_2}{2!}y^n\,x^2 \ldots.$

$\therefore$ Term independent of y in $e^y(1-\frac{x}{y})^n$ is

$e^x\{1 - \frac{{}^{n+1}C_1}{1!}x + \frac{{}^{n+2}C_2 . x^2}{2!} - \ \ldots.\}$

$= e^x\{1 - \frac{(n+1)}{(1!)^2}x + \frac{(n+2)(n+1)}{(2!)^2}x^2 - \ \ldots.\}.$

Hence the required result.

Exercises

1. Show that , if n is a positive integer

   $n\,.\,1^{n+1} - \frac{n(n-1)}{2!}.2^{n+1} + \frac{n(n-1)(n-2)}{3!}.3^{n+1} - \ldots. = (-1)^n.n.\frac{(n+1)!}{2}.$

2. Find the coefficient of x^r in the expansion of $\frac{e^{nx}-1}{1-e^{-x}}$, n being a positive integer and find the values of

(1) $1^2 + 2^2 + 3^2 + \ldots.. + n^2$

(2) $1^3 + 2^3 + 3^3 + \ldots.. + n^3$

(3) $1^4 + 2^4 + \ldots.. + n^4$

3. By means of the identity $e^{x^2+\frac{1}{x^2}+2} = e^{(x+\frac{1}{x})^2}$ show that

$$e^2\{ 1 + \frac{1}{(1!)^2} + \frac{1}{(2!)^2} + \frac{1}{(3!)^2} \ldots.\} = 1 + \frac{2!}{(1!)^3} + \frac{4!}{(2!)^3} + \frac{6!}{(3!)^3} + \ldots.$$

[Left side = term independent of x in $e^2 . e^{x^2} . e^{x^{\frac{1}{2}}}$

$$e^{(x+\frac{1}{x})^2} = 1 + \frac{(x+\frac{1}{x})^2}{1!} + \frac{(x+\frac{1}{x})^4}{2!} + \frac{(x+\frac{1}{x})^6}{4!} + \ldots\ldots$$

Term independent of x in the above expansion

$$= 1 + \frac{{}_2C_1}{1!} + \frac{{}_4C_2}{2!} + \frac{{}_6C_3}{3!} + \ldots..]$$

Answer : 2(1).$\frac{n(n+1)(2n+1)}{6}$,(2).$\frac{n^2(n+1)^2}{4}$,(3).$\frac{n(n+1)(6n^3+9n^2+n-1)}{60}$.

Extra problems

1. Find the coefficient of x^n in $\frac{a+be^x+ce^{2x}}{e^{3x}}$.

Solution.

$$\frac{a+be^x+ce^{2x}}{e^{3x}} = (a + be^x + ce^{2x})e^{-3x}$$

$$= ae^{-3x} + be^{-2x} + ce^{-x}$$

$$= a\left(1 - \frac{(3x)}{1!} + \frac{(3x)^2}{2!} - \cdots + \frac{(-1)^n(3x)^n}{n!} + \cdots\right) + b\left(1 - \frac{(2x)}{1!} + \frac{(2x)^2}{2!} - \cdots + \frac{(-1)^n(2x)^n}{n!} + \cdots\right) + c\left(1 - \frac{(x)}{1!} + \frac{(x)^2}{2!} - \cdots + \frac{(-1)^n(x)^n}{n!} + \cdots\right)$$

∴ coefficient of x^n in $\frac{a+be^x+ce^{2x}}{e^{3x}}$ is = $\frac{(-1)^n}{n!}[a3^n + b2^n + c]$.

2. What is the coefficient of x^n in the expansion of $(1 + x)e^{1+x}$ in ascending powers of x.

Solution.

$(1 + x)e^{1+x} = (1 + x)\, e\, e^x$

$$= e(1 + x)\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} \ldots\right]$$

Therefore coefficient of x^n in $(1 + x)e^{1+x}$ is $= e\left[\frac{1}{n!} + \frac{1}{(n-1)!}\right]$

$= e\left[\frac{1}{n!}+\frac{n}{n!}\right] = \frac{e}{n!}(1+n)$.

3. Prove that $\log 2 - \frac{(\log 2)^2}{2!}+\frac{(\log 2)^3}{3!}-\cdots=\frac{1}{2}$.

Solution.

Put log 2 = y.

Therefore L.H.S = $y - \frac{y^2}{2!}+\frac{y^3}{3!}-\cdots$

$= -\left[-\frac{y}{1}+\frac{y^2}{2!}-\frac{y^3}{3!}+\cdots\right]$

$= -(e^{-y}-1) = 1-e^{-\log 2} = 1-\frac{1}{2}=\frac{1}{2}$.

4. Prove that $\frac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{1+\frac{1}{3!}+\frac{1}{5!}+\cdots}=\frac{e-1}{e+1}$

Solution.

L.H.S = $\frac{\frac{1}{2}(e+e^{-1})}{\frac{1}{2}(e-e^{-1})} = \frac{e^2+1-2e}{e^2-1}$

$= \frac{(e-1)^2}{(e+1)(e-1)}=\frac{e-1}{e+1}$.

5. Show that if a > 1. $S = 1+\frac{1+a}{2!}+\frac{1+a+a^2}{3!}+\cdots=\frac{e^a-e}{a-1}$.

Solution.

n^{th} term $T_n = \frac{1+a+a^2+\cdots+a^{n-1}}{n!}=\frac{a^n}{n!(a-1)}$.

Therefore $T_n = \left(\frac{1}{a-1}\right)\left[\frac{a^n}{n!}-\frac{1}{n!}\right]$ .... (1)

Putting n = 1, 2, 3, ... in (1) we get

$T_1 = \left(\frac{1}{a-1}\right)\left[\frac{a}{1!}-\frac{1}{1!}\right]$

$T_2 = \left(\frac{1}{a-1}\right)\left[\frac{a^2}{2!}-\frac{1}{2!}\right]$

$T_3 = \left(\frac{1}{a-1}\right)\left[\frac{a^3}{3!}-\frac{1}{3!}\right]$

... ...

... ...

Adding we get

$S = \left(\frac{1}{a-1}\right)\left[\left(\frac{a}{1!}+\frac{a^2}{2!}+..\right)-\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)\right]$

$= \left(\frac{1}{a-1}\right)[(e^a-1)-(e-1)] = \frac{e^a-e}{a-1}$.

6. Prove that $S = 1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\cdots=\frac{3e}{2}$.

Solution.

n^{th} term $T_n = \frac{1+2+\cdots+n}{n!}$

$= \frac{n(n+1)}{2n!}=\frac{n+1}{2(n-1)!}$.

Let n+ 1 =A +B(n – 1).

Putting n = 1 and n = 0 we get A = 2; B = 1.

Therefore $T_n = \frac{2+(n-1)}{2(n-1)!}$

Therefore $T_n = \frac{1}{(n-1)!} + \frac{1}{2(n-2)!}$ ....(1)

Putting n = 1, 2, 3, ... in (1) we get

$T_1 = 1$

$T_2 = \frac{1}{1!} + \frac{1}{2!}$

$T_3 = \frac{1}{2!} + \left(\frac{1}{2}\right)\frac{1}{1!}$

$T_4 = \frac{1}{3!} + \left(\frac{1}{2}\right)\frac{1}{2!}$

... ...

... ...

Adding we get S = $\left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] + \frac{1}{2}\left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$

= $e + \frac{1}{2}e = \frac{3e}{2}$.

7. Find $S = \sum_{n=1}^{\infty} \frac{n-1}{(n+2)n!} x^n$.

Solution.

Here the n^{th} term $T_n = \frac{n-1}{(n+2)n!} x^n$

$= \frac{n^2-1}{(n+2)!} x^n$.

Now, let n2 – 1 = A + B(n + 2) + C(n +2)(n +1).

We get A = 3, B = - 3, C = 1

Therefore $T_n = \frac{3}{(n+2)!}x^n - \frac{3}{(n+1)!}x^n + \frac{1}{n!}x^n$ ....(1)

Putting n = 1, 2, 3, ... in (1)we get

$T_1 = \frac{3}{3!}x - \frac{3}{2!}x + \frac{1}{1!}x$

$T_2 = \frac{3}{4!}x^2 - \frac{3}{3!}x^2 + \frac{1}{2!}x^2$

$T_3 = \frac{3}{5!}x^3 - \frac{3}{4!}x^3 + \frac{1}{3!}x^3$

... ...

... ...

Adding we get

$S = 3\left[\frac{x}{3!} + \frac{x^2}{4!} + \cdots\right] - 3\left[\frac{x}{2!} + \frac{x^2}{3!} + \cdots\right] + \left[\frac{x}{1!} + \frac{x^2}{2!} + \cdots\right]$

$= \frac{3}{x^2}\left[\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right] - \frac{3}{x}\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] + \left[\frac{x}{1!} + \frac{x^2}{2!} + \cdots\right]$

$= \frac{3}{x^2}\left(e^x - \frac{x^2}{2!} - \frac{x}{1!} - 1\right) - \frac{3}{x}\left(-\frac{x}{1!} - 1\right) + (e^x - 1)$

$= \frac{3}{x^2} e^x - \frac{3}{2} - \frac{3}{x} - \frac{3}{x^2} - \frac{3}{x} e^x + 3 + \frac{3}{x} + e^x - 1$

$= e^x \left(\frac{3}{x^2} - \frac{3}{x} + 1\right) - \frac{3}{x^2} + \frac{1}{2} = e^x \left(\frac{3-3x+x^2}{x^2}\right) + \left(\frac{x^2-6}{2x^2}\right).$

$= \frac{2e^x(x^2-3x+3)+(x^2-6)}{2x^2}.$

8. Show that $\frac{1^2 2^2}{1!} + \frac{2^2 3^2}{2!} + \frac{3^2 4^2}{3!} + \cdots$ = 27e.

Solution.

n^{th} term $T_n = \frac{n^2(n+1)^2}{n!} = \frac{n(n+1)^2}{(n-1)!}$

let $n(n + 1)^2 = A + B(n - 1) + C(n - 1)(n - 2) + D (n - 1)(n - 2)(n - 3)$

we get A = 4;B = 14; C = 8; D = 1.

Therefore $T_n = \frac{4}{(n-1)!} + \frac{14}{(n-2)!} + \frac{8}{(n-3)!} + \frac{1}{(n-4)!}.$

$T_1 = \frac{4}{1}$

$T_2 = \frac{4}{1!} + 14$

$T_3 = \frac{4}{2!} + \frac{14}{1!} + 8$

$T_4 = \frac{4}{3!} + \frac{14}{2!} + \frac{8}{1!} + 1$

$T_5 = \frac{4}{4!} + \frac{14}{3!} + \frac{8}{2!} + \frac{1}{1!}$

… … … … …

… … … … …

Adding we get

Therefore $S = 4\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) + 14\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) + 8\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) + \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right)$

= 4e +14e +8e + e = 27e.

Logarithmic series

$$\log(1 + x) = x - \frac{x^2}{2!} + \frac{1.2}{3!} x^3 - \frac{1.2.3}{4!} x^4 \ \ldots..$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots..$$

Modification of the logarithmic series.

If $-1 < x < 1$, we have

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ \ldots.. + (-1)^{n-1} \frac{x^n}{n} + \ldots \qquad \ldots\ldots\ldots(1)$$

It is convenient to remember the form of the series in the case in which x is negative.

Thus

$$\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \ \ldots\ldots$$

$$= -\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right)$$

i.e., $-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$ ....(2)

Adding the series (1) and (2),

$$\log(1+x) - \log(1-x) = 2x + 2.\frac{1}{3}x^3 + 2.\frac{1}{5}x^5 + \ldots\ldots$$

i.e., $\log\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{2} + \frac{x^5}{3} + \cdots\right)$

$$\log(1+x) + \log(1-x) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots\right)$$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Using the different forms of the logarithmic series we can find the sums of the certain series. The following examples will illustrate the methods of such summation.

Example 1. Show that if $x > 0$. $\log x = \frac{x-1}{x+1} + \frac{1}{2}.\frac{x^2-1}{(x+1)^2} + \frac{1}{3}.\frac{x^3-1}{(x+1)^3} + \ldots..$

Solution.

$$\text{R.H.S.} = \frac{x}{x+1} + \frac{1}{2}.\left(\frac{x}{x+1}\right)^2 + \frac{1}{3}.\left(\frac{x}{x+1}\right)^3 \ldots.. - \left\{\frac{1}{x+1} + \frac{1}{2}.\frac{1}{(x+1)^2} + \frac{1}{3}.\frac{1}{(x+1)^3} \ldots\right\}$$

$$= -\log\left(1 - \frac{x}{x+1}\right) + \log\left(1 - \frac{x}{x+1}\right)$$

$$= -\log\frac{1}{x+1} + \log\frac{x}{x+1}$$

$$= \log\left\{\left(\frac{x}{x+1}\right) + \frac{1}{x+1}\right\}$$

$$= \log x \ .$$

The expansion is valid when

$\left|\frac{x}{x+1}\right| < 1$ and $\left|\frac{1}{x+1}\right| < 1$, $\left|\frac{x}{x+1}\right|$ is always less than 1.

When $\left|\frac{1}{x+1}\right| < 1$, $|x+1| > 1$, i.e., $|x| > 0$

∴ When $x > 0$, the expansion is valid .

Example 2. Show that $\log\sqrt{12} = 1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3} + \ldots.$

Solution.

Right side expression can be written as

$$\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4^2} + \frac{1}{6}\cdot\frac{1}{4^3} + \ldots. + 1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4^2} + \frac{1}{7}\cdot\frac{1}{4^3} + \ldots..$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^4 + \frac{1}{6}\cdot\left(\frac{1}{2}\right)^6 + \ldots.. + 1 + \frac{1}{3}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{5}\cdot\left(\frac{1}{2}\right)^4 + \frac{1}{7}\cdot\left(\frac{1}{2}\right)^6 + \ldots.$$

$$= \frac{1}{2}\cdot x^2 + \frac{1}{4}x^4 + \frac{1}{6}x^6 + \ldots.. + 1 + \frac{1}{3}x^2 + \frac{1}{5}x^4 + \frac{1}{7}x^6 + \ldots. \text{ When } x = \frac{1}{2}$$

$$= \frac{1}{2}\left\{ x^2 + \frac{1}{2}\cdot x^4 + \frac{1}{3}x^6 + \ldots..\right\} + \frac{1}{x}\left\{ x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \ldots\right\}$$

$$= -\frac{1}{2}\log(1-x^2) + \frac{1}{2x}\log\frac{1+x}{1-x}\ .$$

∴ The series $= -\frac{1}{2}\log\left(1-\frac{1}{4}\right) + \log\frac{1+\frac{1}{2}}{1-\frac{1}{2}}$, since $x = \frac{1}{2}$.

$$= -\frac{1}{2}\log\frac{3}{4} + \log 3$$

$$= \frac{1}{2}\log 9 - \frac{1}{2}\log\frac{3}{4}$$

$$= \frac{1}{2}\log\left(\frac{9.4}{3}\right)$$

$$= \frac{1}{2}\log 12$$

$$= \log\sqrt{12}.$$

Example 3. If a , b , c denote three consecutive integers, show that

$$\log_e b = \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2ac+1} + \frac{1}{3}\cdot\frac{1}{(2ac+1)^3} + \ldots.$$

Solution.

$$\text{Right side} = \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log_e \frac{1+\frac{1}{2ac+1}}{1-\frac{1}{2ac+1}}$$

$$= \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log_e \frac{2ac+1}{2ac}$$

$$= \frac{1}{2}\log(ac) + \frac{1}{2}\log\frac{ac+1}{ac}$$

$$= \frac{1}{2}\log ac\,.\,\frac{ac+1}{ac}$$

$$= \frac{1}{2}\log(ac+1).$$

If a , b , c denote three consecutive integers then $b = a + 1$ and $b = c - 1$

$\therefore\ a = b - 1$; $c = b + 1$.

$\therefore\ ac = b^2 - 1$, i.e., $ac + 1 = b^2$.

$\therefore\ \frac{1}{2}\log(ac+1) = \frac{1}{2}\log(b^2) = \log b.$

Exercises

1. Show that

$$\log\frac{a+x}{a-x} = \frac{2ax}{a^2+x^2} + \frac{1}{3}\cdot\left(\frac{2ax}{a^2+x^2}\right)^3 + \frac{1}{5}\cdot\left(\frac{2ax}{a^2+x^2}\right)^5 + \ldots..$$

2. Sum the series $\frac{1}{2x-1} + \frac{1}{3}\cdot\frac{1}{(2x-1)^3} + \frac{1}{5}\cdot\frac{1}{5(2x-1)^5} + \ldots.$

3. Show that when $-1 < x < \frac{1}{3}$

$$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right) = \frac{2x}{1-x} - \frac{1}{2}\cdot\left(\frac{2x}{1-x}\right)^2 + \frac{1}{3}\cdot\left(\frac{2x}{1-x}\right)^3 \ldots.$$

4. Show that

$\log(x+2h) = 2\log(x+h) - \log(x) - \left\{\frac{h^2}{(x+h)^3} + \frac{h^4}{2(x+h)^3} + \frac{h^6}{3(x+h)^3} + \cdots\right\}.$

5. Show that

$\log_e\left(1+\frac{1}{n}\right)^2 = 1 - \frac{1}{2(n+1)} - \frac{1}{2.3(n+1)^2} - \frac{1}{3.4(n+1)^3} \ldots\ldots \infty.$

6. Show that $\log_e 3 = 1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \ldots\ldots$

7. Sum the series $(1+\frac{1}{2}) + \left(\frac{1}{3}+\frac{1}{4}\right)\frac{1}{9} + \left(\frac{1}{5}+\frac{1}{6}\right)\frac{1}{9^2} + \ldots\ldots$ to infinity.

8. Sum to infinity the series $\sum\left(\frac{1}{2n+1}+\frac{1}{(2n)!}\right)x^{2n+1}$, $(x^2 < 1)$.

9. Prove that $\sum_1^\infty \frac{1}{2n-1}\left(\frac{1}{9^{n-1}}+\frac{1}{9^{2n-1}}\right) = \frac{1}{2}\log_e 10.$

Answer : 2. $\frac{1}{2}log\left(\frac{x}{x-1}\right)$, 7. 9log 3 – 12log 2, 8. $\frac{1}{2}\left[log\frac{1+x}{1-x} + x(e^x + e^{-x})\right]$.

Series which can be summed up by the logarithmic series.

We can split the general term into partial fractions and using the result

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$ We can sum certain series. The following examples will illustrate the method.

Example 1. Sum the series $\sum_{n=1}^\infty \frac{1}{(2n-1)2n(2n+1)}$.

Solution.

Let S be the sum of the series and u_n be the n^{th} term.

Then $u_n = \frac{1}{2}.\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2}.\frac{1}{2n+1}$

$\therefore\ u_1 = \frac{1}{2}.\frac{1}{1} - \frac{1}{2} + \frac{1}{2}.\frac{1}{3}$

$u_2 = \frac{1}{2}.\frac{1}{3} - \frac{1}{4} + \frac{1}{2}.\frac{1}{5}$

$u_3 = \frac{1}{2}.\frac{1}{5} - \frac{1}{6} + \frac{1}{2}.\frac{1}{7}$

…………………………………………

…………………………………………

Adding the last fraction of a term with the first fraction of the next term, we get

$$S = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \ldots..$$

$$= -\frac{1}{2} + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots.$$

$$= -\frac{1}{2} + \log 2.$$

Example 2. Show that $\frac{5}{1.2.3} + \frac{7}{3.4.5} + \frac{9}{5.6.7} + \ldots. \infty = 3 \log 2 - 1.$

Solution.

Let S be the sum of the series and u_n be the n^{th} term of the series.

Then $u_n = \frac{2n+3}{(2n-1)(2n+1)}$.

Splitting u_n into partial fractions, we get

$$u_n = 2 . \frac{1}{2n-1} - 3 . \frac{1}{2n} + 1 . \frac{1}{2n+1}$$

Giving values 1 , 2 , 3 , …. in u_n , we have

$$u_1 = 2 . \frac{1}{1} - 3 . \frac{1}{2} + 1 . \frac{1}{3}$$

$$u_2 = 2 . \frac{1}{3} - 3 . \frac{1}{4} + 1 . \frac{1}{5}$$

$$u_3 = 2 . \frac{1}{5} - 3 . \frac{1}{6} + 1 . \frac{1}{7}$$

…………………………………………

…………………………………………

$$\therefore \ S = 2 - 3 . \frac{1}{2} + 3 . \frac{1}{3} - 3 . \frac{1}{4} + 3 . \frac{1}{5} \ldots.$$

$$= 2 + 3(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5})$$

$= 2 + 3(1 - \frac{1}{2} + \frac{1}{3} \dots\dots - 1)$

$= 2 + 3 (\log 2 - 1)$

$= -1 + 3 \log 2.$

Exercises

Show that the sum of the series to infinity

1. $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{3.6} + \dots. = \log 2$
2. $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \dots. = 2 - \log 2.$
3. $\frac{1}{1.2.3} + \frac{5}{3.4.5} + \frac{9}{5.6.7} + \frac{13}{7.8.9} + \dots. = \frac{5}{2} - 3 \log 2.$
4. $\frac{1}{2.3.4} + \frac{5}{4.5.6} + \frac{9}{6.7.8} + \dots. = \frac{3}{4} - \log 2$

If k is a positive integer and $|x| < 1$, then

$$\sum_{n=1}^{\infty} \frac{x^2}{n+k} = \frac{x}{1+k} + \frac{x^2}{2+k} + \frac{x^3}{3+k} + \frac{x^4}{4+k} + \dots\dots$$

$$= \frac{1}{x^k}\left(\frac{x^{k+1}}{k+1} + \frac{x^{k+2}}{k+2} + \frac{x^{k+3}}{k+3} + \cdots \infty\right)$$

$$= \frac{1}{x^k} \{ x + \frac{x^2}{2} + \dots. + \frac{x^k}{k} + \frac{x^{k+1}}{k+1} + \frac{x^{k+2}}{k+2} + \frac{x^{k+3}}{k+3} + \dots \infty$$

$$-(x + \frac{x^2}{2} + \dots. + \frac{x^k}{k}) \}$$

$$= \frac{1}{x^k} \{ - \log (1 - x) - (x + \frac{x^2}{2} + \dots. + \frac{x^k}{k}) \}$$

$$= -\frac{1}{x^k} \{ \log (1 - x) + x + \frac{x^2}{2} + \dots. + \frac{x^k}{k} \}$$

Similarly $\sum_{n=1}^{\infty} \frac{x^n}{n+1} = -\frac{1}{x} \{ \log (1 - x) + x\}$

$$\sum_{n=1}^{\infty} \frac{x^n}{n+2} = -\frac{1}{x^2} \{ \log (1 - x) + x + \frac{x^2}{2} \}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n+3} = -\frac{1}{x^3} \{ \log (1 - x) + x + \frac{x^2}{2} + \frac{x^3}{3} \}$$

Using these result we can sum certain series. The following examples will illustrate the method.

Example 1. Sum the series $\sum_{n=1}^{\infty} \frac{n^3+n^2+1}{n(n+2)} x^n$ when $|x| < 1$.

Solution.

Split $\frac{n^3+n^2+1}{n(n+2)}$ into partial fractions.

We have $S = \sum_{n=1}^{\infty}\{ (n-1) + \frac{1}{2} \cdot \frac{1}{n} + \frac{3}{2} \cdot \frac{1}{n+2} \} x^n$

$= \sum_{n=1}^{\infty}\{ (n-1)x^n + \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^n}{n} + \frac{3}{2} \sum_{n=1}^{\infty} \frac{x^n}{n+2}$.

$\sum_{n=1}^{\infty}\{ (n-1)x^n = x^2 + 2x^3 + 3x^4 + \ldots.. \infty$

$= x^2 (1 + 2x + 3x^2 + \ldots. \infty)$

$= x^2 (1-x)^2 = \frac{x^2}{(1-x)^2}$.

$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log (1-x)$.

$\sum_{n=1}^{\infty} \frac{x^n}{n+2} = -\frac{1}{x^2} \{ \log (1-x) + x + \frac{x^2}{2} \}$

$\therefore$ $S = \frac{x^2}{(1-x)^2} - \frac{1}{2} \log (1-x) - \frac{3}{2x^2} \{ \log (1-x) + x + \frac{x^2}{2} \}$.

Example 2. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n(n+1)(n+2)}$.

Solution.

$\frac{1}{n(n+1)(n+2)} = \frac{1}{2} \cdot \frac{1}{n} - \frac{1}{n+1} + \frac{1}{2} \cdot \frac{1}{n+2}$

Let S be the sum of the series

$S = \sum_{1}^{\infty} \left(\frac{1}{2} \cdot \frac{1}{n} - \frac{1}{n+1} + \frac{1}{2} \cdot \frac{1}{n+2}\right) (-1)^{n+1} x^n$

$$= \frac{1}{2} \cdot \sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n} - \sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n+1} + \frac{1}{2}\sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n+2}.$$

We have

$$\sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n} = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} \ldots.. = \log(1+x)$$

$$\sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n+1} = \frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} \ldots\ldots = \frac{1}{x}\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} \ldots.\right)$$

$$= \frac{1}{x}\{-\log(1+x) + x\}$$

$$\sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n+2} = \frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} \ldots\ldots = \frac{1}{x^2}\left\{\frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} \ldots.\right\}$$

$$= \frac{1}{x^2}\left\{\log(1+x) - x + \frac{x^2}{2}\right\}.$$

$$\therefore\ S = \frac{1}{2}\log(1+x) - \frac{1}{x}\{-\log(1+x) + x\} + \frac{1}{2x^2}\left\{\log(1+x) - x + \frac{x^2}{2}\right\}$$

$$= \frac{1}{2}\log(1+x)\left(1 + \frac{2}{x} + \frac{1}{x^2}\right) - \left(\frac{3}{4} + \frac{1}{2x}\right).$$

Exercises

1. Prove that the sum of the infinite series whose n^{th} term is $\frac{1}{n(n+1)} \cdot \frac{1}{2^n}$ is $1 - \log 2$.
2. Sum the series

   (1) $\sum_1^\infty \frac{n^2+1}{n(n+2)} x^n$.

   (2) $\sum_1^\infty \frac{(n+1)^3}{n(n+3)} x^n$.

   (3) $\sum_1^\infty \frac{n^2}{(n+1)(n+2)} x^n$.
3. Show that

   (1) $\frac{3}{1.2.2} - \frac{4}{2.3.2^2} + \frac{5}{3.4.2^3} - \ldots. = 4\log\frac{3}{2} - 1$.
4. Show that

   (1) $\sum_{r=1}^\infty \frac{4r-1}{2r(2r-1)} \cdot \frac{1}{3^{2r}} = \log 3 - \frac{4}{3}\log 2$.

(2) $\sum_{r=1}^{\infty} \frac{10r+1}{2r(2r-1)(2r+1)} \cdot \frac{1}{2^{2r}} = 2 - \log 2 - \frac{3}{4} \log 3.$

Answer : 2(1).$\frac{5-x^2}{2x^2}\log(1-x) + \frac{(x^2+5x-10)}{4x(x-1)}$,(2).$\frac{x}{(1-x)^2} - \frac{4}{9}(2x^3 + 3x^2 + 6x) - 3\log(1-x)$,(3).$\frac{9-4x^3}{12x^2}\log(1-x) + \frac{6x^3-x^2-3x+6}{8x(1-x)}$.

Calculation of logarithms by means of the logarithmic series.

The direct calculation of logarithms by means of the series

$$\text{Log}\,(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\ldots \infty$$

is somewhat tedious , since the series is slowly convergent, i.e., very many terms of the series have to be calculated before a given degree of approximation is attained.

The calculation is usually carried out in practice as follows.

We have proved that

$$\log_e \frac{1+x}{1-x} = 2\left\{ x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right\}$$

When $-1 < x < 1$.

Let $y = \frac{1+x}{1-x}$, i.e., $x = \frac{y-1}{y+1}$.

$$\therefore \log_e y = 2\left\{\frac{y-1}{y+1} + \frac{1}{3}\cdot\left(\frac{y-1}{y+1}\right)^3 + \frac{1}{5}\cdot\left(\frac{y-1}{y+1}\right)^5 + \cdots\right\}$$

Where y lies between 0 and $+\infty$.

Put $y = \frac{p}{q}$ in this series where p and q are positive integers.

$$\therefore \log_e p - \log_e q = 2\left\{ \left(\frac{p-q}{p+q}\right) + \frac{1}{3}\cdot\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\cdot\left(\frac{p-q}{p+q}\right)^5 + \ldots\ldots\right.$$

Now if p and q be fairly large and differ little in value, i.e., $(p-q)$ is small, the above series converges rapidly to the limits, since the terms become small quickly.

Example. Evaluate log 2 to 5 places of decimals.

Solution.

Put $p = 2$, $q = 1$.

$\therefore$ $\log_e 2 - \log_e 1 = 2 \cdot \left(\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{5^3} + \cdots\right)$

$\log_e 1 = 0.$

$\frac{1}{3} = 0.333,333,3$ $\frac{1}{3^3} = 0.037,037$ $\frac{1}{3} \cdot \frac{1}{3^3} = 0.012,345,7$ $\frac{1}{3^5} = 0.004,115,2$

$\frac{1}{5} \cdot \frac{1}{3^5} = 0.000,832,0$ $\frac{1}{3^7} = 0.000,457,2$ $\frac{1}{7} \cdot \frac{1}{3^7} = 0.000,055,3$ $\frac{1}{3^9} = 0.000,050,8$

$\frac{1}{9} \cdot \frac{1}{3^9} = 0.000,005,6$ $\frac{1}{3^{11}} = 0.000,005,6$ $\frac{1}{11} \cdot \frac{1}{3^{11}} = 0.000,000,5$

$\therefore$ Sum of the first 6 terms is 2 (0.346,573,4) approximately

i.e., 0.693,146,8

$\therefore$ log 2 = 0.69315 to 5 places of decimals.

We can calculate the error involved in taking only the first six terms.

The difference between log 2 and the sum of the first six terms.

$$= 2\left\{\frac{1}{13} \cdot \frac{1}{3^{13}} + \frac{1}{15} \cdot \frac{1}{5^{15}} + \cdots\right\}$$

$$< \frac{2}{13}\left\{\frac{1}{3^{13}} + \frac{1}{3^{15}} + \cdots \infty\right\}$$

$$< \frac{2}{13} \cdot \frac{1}{3^{13}} \cdot \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots \infty\right)$$

$$< \frac{2}{13} \cdot \frac{1}{3^{13}} \cdot \frac{1}{1 - \frac{1}{3^2}}$$

$$< \frac{2}{13} \cdot \frac{1}{3^{13}} \cdot \frac{9}{8}$$

$$< \frac{1}{13} \cdot \frac{1}{3^{11}} \cdot \frac{1}{4}$$

$< \frac{1}{52}$. (0.0000056)

< 0.0000011.

Hence if we take log 2 = 0.69315, there is no error until the 6th place of decimals.

By means of this series by putting $p = 5$, $q = 4$, $\log_e 3$ can be calculated.

By putting $p = 5$, $q = 4$, $\log_e 3$ can be calculated.

Similarly we can calculated logarithms of numbers.

The application of the exponential and logarithmic series to limits and approximations.

The application is shown in the following examples:

Example 1. Evaluate $Lt_{x\to 0} \frac{e^x - e^{-x}}{\log(1+x)}$.

Solution.

$$Lt_{x\to 0} \frac{e^x - e^{-x}}{\log(1+x)} .$$

$$= Lt_{x\to 0} \frac{(1 + x + \frac{x^2}{2!} \ldots) - (1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots)}{x - \frac{x^2}{2!} + \frac{x^3}{3!} \ldots}$$

$$= Lt_{x\to 0} \frac{2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots}{x - \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

$$= Lt_{x\to 0} \frac{2 + \frac{2x}{3!} + \frac{2x^4}{5!} + \cdots}{1 - \frac{x}{2!} + \frac{x^2}{3!} - \cdots}$$

= 2.

Example 2. Evaluate $Lt_{n\to\infty} \left(1 + \frac{3}{n^2} + \frac{5}{n^3}\right)^{n^2+7n}$

Solution.

Let the value of the limit be A.

$\therefore$ A = $Lt_{n\to\infty}\ (1+\frac{3}{n^2}+\frac{5}{n^3})^{n^2+7n}$

Taking logarithms on both sides, we have

log A = $Lt_{n\to\infty}\ (1+\frac{3}{n^2}+\frac{5}{n^3})^{n^2+7n}$

$= Lt_{n\to\infty}\ (n^2+7n)(1+\frac{3}{n^2}+\frac{5}{n^3})$

$= Lt_{n\to\infty}(n^2+7n)\{\left(\frac{3}{n^2}+\frac{5}{n^3}\right)-\frac{1}{2}\left(\frac{3}{n^2}+\frac{5}{n^3}\right)^2+\frac{1}{3}\left(\frac{3}{n^2}+\frac{5}{n^3}\right)^3-\cdots\}$

$= Lt_{n\to\infty}(n^2+7n)\{\left(\frac{3}{n^2}+\frac{5}{n^3}\right)-\frac{1}{2n^4}\left(3+\frac{5}{n}\right)^2+\frac{1}{3n^6}\left(3+\frac{5}{n}\right)^3\ldots..\}$

$= \underset{n\to\infty}{Lt}\ \{3+\frac{5}{n}+\frac{21}{n}+\frac{35}{n^2}-\frac{1}{2n^2}\left(3+\frac{5}{n}\right)^2-\frac{7}{2n}\left(3+\frac{5}{n}\right)^2+\ldots.\}$

Except the first , all the other term will contain $\frac{1}{n}$ or higher powers of $\frac{1}{n}$.

$\therefore$ log A = 3.

$\therefore$ A = e^3.

Example 3. Prove that, if n is large $\left(n-\frac{1}{3n}\right)\log\frac{n+1}{n-1} = 2+\frac{8}{45n^4}+\ldots.$

and $\left(\frac{n+1}{n-1}\right)^{n-\frac{1}{3n}} = e^2\left(1+\frac{8}{45n^4}+\ \ldots\right)$

Solution.

Let $\left(\frac{n+1}{n-1}\right)^{n-\frac{1}{3n}}$ be A.

$\therefore$ log A = $\left(n-\frac{1}{3n}\right)\log\frac{n+1}{n-1}$

$= \left(n-\frac{1}{3n}\right)\log\frac{1+\frac{1}{n}}{1-\frac{1}{n}}$

$= \left(n-\frac{1}{3n}\right)\{\ \log(1+\frac{1}{n})-\log(1-\frac{1}{n})\ \}$

$$= 2\left(n - \frac{1}{3n}\right)\left\{\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \frac{1}{7n^7} + \ldots.\right\}$$

$$= 2\left\{ 1 + \frac{1}{3n^2} + \frac{1}{5n^4} + \frac{1}{7n^6} + \ldots. - \frac{1}{3n^2} - \frac{1}{9n^4} \ldots.\right\}$$

$$= 2\left\{1 + \frac{4}{45n^4} + \ldots.\right\}$$

$$= 2 + \frac{8}{45n^4}$$

$$\therefore \quad A = e^{2+\frac{8}{45n^4}}$$

$$= e^2.\left\{1 + \frac{8}{45n^4} + \ldots\right\}$$

Example 4. Show that if $e^x = 1 + xe^{yx}$, where x^3 and higher powers of x can be neglected,

$$y = \frac{1}{2!} + \frac{1}{4!}$$

Solution.

Now $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots..$

$$\therefore \quad e^x - 1 = x\left\{1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right\}$$

$$\therefore \quad xe^{yx} = x\left\{1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right\}$$

$$\therefore \quad e^{yx} = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots.$$

Taking logarithms on both sides, we have

$$yx = \log\left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots.\right)$$

$$= \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^4}{4!} + \ldots.\right) - \frac{1}{2}\left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots.\right)^2$$

$$+\frac{1}{3}\left(\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\ \dots\right)^3$$

$= \frac{x}{2} + \frac{x^2}{24}$ + terms in x^4 and higher powers of *x*.

Hence $y = \frac{1}{2} + \frac{x}{24}$.

Exercises

1. Evaluate $\lim_{x\to 0} \frac{e^x - \log(e+ex)}{x^2}$.
2. Evaluate $\lim_{x\to 0} \frac{e^x - \log_e(1+x)(1+2x)}{5x^3}$.
3. Find $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$.
4. Find the limit as $x \to 1$ of $\frac{\log x}{x^2-3x+2}$.
5. Evaluate $\lim_{x\to 0} \frac{(2+x)\log(1+x)+(2-x)\log(1-x)}{x^4}$.
6. Evaluate $\lim_{n\to\infty} (1 + \frac{3}{n^2} + \frac{1}{n^3})^{n^2}$.
7. Find the value, when *x* tends to the limit 1 of the expression $\log(x^{5/2} - 1) - \log(x^{3/2} - 1)$.
8. Show that when *x* is small , log $\{(1+x)^{1/3} + (1-x)^{1/3}$ is approximately equal to log 2 $- \frac{x^2}{9}$.
9. By using the fact that $(1+\frac{x}{n})^n = e^{x\log(1+\frac{x}{n})}$ prove that $(1+\frac{x}{n})^n + (1-\frac{x}{n})^{-n} = 2e^x\ \{1 + \frac{1}{n^2}\left(\frac{x^2}{3}+\frac{x^4}{8}\right)\}$.

Answer : 1.2, 2. $\frac{1}{10}$, 3. $\frac{3}{2}$, 4 – 1, 5. .$-\frac{1}{3}$,6. e^3, 7 $log\left(\frac{5}{3}\right)$.

Extra problems.

1. Show that $\left[\frac{a-b}{a}\right] + \frac{1}{2}\left[\frac{a-b}{a}\right]^2 + \frac{1}{3}\left[\frac{a-b}{a}\right]^3 + \dots = \log_e a - \log_e b$.

   Solution.

   Put $\frac{a-b}{a} = x$.

Therefore L.H.S = x + $\frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$

= - log(1 – x)

= - log$\left(1 - \frac{a-b}{a}\right)$ = - log$\left(\frac{b}{a}\right)$ = log$\left(\frac{a}{b}\right)$

= log a – log b.

= R.H.S.

2. Prove that log $\sqrt{\frac{n+1}{n}} = \left(\frac{1}{2n+1}\right) + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \cdots$

Solution.

Let $\frac{1}{2n+1}$ = x.

Therefore R.H.S = $\frac{1}{2}log\left(\frac{1+x}{1-x}\right) = \frac{1}{2}log\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n-1}}\right)$

= $\frac{1}{2}log\left(\frac{2n+2}{2n}\right) = \frac{1}{2}log\left(\frac{n+1}{n}\right)$

= log $\sqrt{\frac{n+1}{n}}$.

= L.H.S.

3. Show that $\frac{3}{10}\left[log10 + \frac{1}{2^7} + \frac{1}{2}.\frac{1}{2^{14}} + \frac{1}{3}.\frac{3^2}{2^{21}} + \cdots\right]$ = log 2.

Solution.

L.H.S = $\frac{1}{10}\left[3log10 + \left(\frac{3}{2^7}\right) + \frac{1}{2}.\left(\frac{3}{2^7}\right)^2 + \frac{1}{2}.\left(\frac{3}{2^7}\right)^3 + \cdots\right]$

= $\frac{1}{10}\left[log1000 - log\left(1 - \frac{3}{2^7}\right)\right]$

= $\frac{1}{10}\left[log1000 - log\left(\frac{125}{2^7}\right)\right]$

= $\frac{1}{10}\,log\left(\frac{1000 \times 2^7}{125}\right)$

= $\frac{1}{10}\,log2^{10}$ = log 2 = R.H.S.

4. Sum to infinity the series $\left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right)\left(\frac{1}{9}\right) + \left(\frac{1}{5} + \frac{1}{6}\right)\left(\frac{1}{9^2}\right) + \cdots$

Solution.

$$\left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right)\left(\frac{1}{9}\right) + \left(\frac{1}{5} + \frac{1}{6}\right)\left(\frac{1}{9^2}\right) + \cdots$$

= $\left[1 + \left(\frac{1}{3} - \frac{1}{9}\right) + \frac{1}{5}\left(\frac{1}{9^2}\right) + \cdots\right] + \left[\frac{1}{2} + \frac{1}{4}\left(\frac{1}{9}\right) + \frac{1}{6}\left(\frac{1}{9^2}\right) + \cdots\right]$

= $3\left[\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \cdots\right] + \frac{9}{2}\left[\frac{1}{9} + \frac{1}{2}\left(\frac{1}{9}\right)^2 + \frac{1}{3}\left(\frac{1}{9}\right)^3 + \cdots\right]$

= $3\left[\frac{1}{2}log\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)\right] - \frac{9}{2}log\left(1 - \frac{1}{9}\right)$

$= \frac{3}{2}$log - $\frac{9}{2}\ log\left(\frac{8}{9}\right) = \frac{3}{2}\left[log2 - 3log\left(\frac{8}{9}\right)\right]$

$= \frac{3}{2}[log2 - 3log8 + 3log9]$

$= \frac{3}{2}[log2 - 9log2 + 6log3]$

$= \frac{3}{2}[6log3 - 8log2]$

= 9log3 – 12log2.

5. Prove that $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots$

Solution.

Put x = $\frac{1}{n+1}$

Then L.H.S = x + $\frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$

= - log(1 – x) = - log$\left(1 - \frac{1}{n+1}\right)$ = - log$\left(\frac{n}{n+1}\right)$

= log$\left(\frac{n+1}{n}\right)$ = log$\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots$

= R.H.S.

6. If y = x - $\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ prove that x = $\frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

Solution.

y = x - $\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ (i.e) y = log(1 + x)

e^y = 1 + x

Therefore x = e^y – 1 = $\left[\frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right] - 1.$

Therefore x = $\frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

7. If x = y - $\frac{y^2}{2!} + \frac{y^3}{3!} - \cdots$ and $|x|$ < 1 show that y = x $+ \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

Solution.

x = y - $\frac{y^2}{2!} + \frac{y^3}{3!} - \cdots$

= - $\left[-\frac{y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \cdots\right]$

= - [e^{-y} – 1]

Thus x = 1 – e^{-y}

e^{-y} = 1 – x

-y = $\log_e$ (1 – x)

y = - $\log_e$(1 – x)

Therefore y = x + $\frac{x^2}{2} + \frac{x^3}{3} + \cdots$.

8. If log (1 – x + x^2) be expanded in ascending powers of x in the form $a_1x + a_2x^2 + a_3x^3$ +........ prove that $a_3 + a_6 + a_9$ +........ = $\frac{2}{3}$ log 2.

Solution.

$\log(1 - x + x^2) = \log\left[\frac{1+x^3}{1+x}\right]$

= log (1+ x^3) – log (1+ x)

$= \left[x^3 - \frac{(x^3)^2}{2} + \cdots + \frac{(-1)^{n-1}(x^3)^n}{n} + \cdots\right] - \left[x - \frac{x^2}{2} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots\right]$

Coefficient of x^{3n} is $a_{3n} = \frac{(-1)^{n-1}}{n} - \frac{(-1)^{3n-1}}{3n}$

$= \frac{(-1)^{n-1}}{n}\left[1 - \frac{1}{3}\right]$

$= (-1)^{n-1}[\frac{2}{3n}]$ ....(1)

Putting n = 1, 2, 3, ... in (1) and adding we get

$a_3 + a_6 + a_9 + \ldots = \frac{2}{3}\left[1 - \frac{1}{2} + \frac{1}{3} - \cdots\right]$

$= \frac{2}{3}\ log_e 2.$

9. Show that if x > 0 log x = $\frac{x-1}{x+1} + \frac{1}{2}\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \cdots$

Solution.

R.H.S = $\left(\frac{x}{x+1}\right) + \frac{1}{2}\left(\frac{x}{x+1}\right)^2 + \frac{1}{3}\left(\frac{x}{x+1}\right)^3 + \cdots$

$+ \left[-\left(\frac{1}{x+1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)^2 - \frac{1}{3}\left(\frac{1}{x+1}\right)^3 - \cdots\right]$

$= -\log\left[1 - \frac{x}{x+1}\right] + \log\left[1 - \frac{1}{x+1}\right]$

$= -\log\left[\frac{1}{x+1}\right] + \log\left[\frac{x}{x+1}\right]$

= log x

= L.H.S.

10. If f(x) = x + $\frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$ where -1 < x < 1.

(i) Represent f(x) as a logarithmic function

(ii) Hence prove $f\left(\frac{2x}{1+x^2}\right)$ = 2f(x)

Solution. (i) For – 1 < x < 1 we have

$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

$\log(1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$

$\log(1 + x) - \log(1 - x) = 2\left[x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots\right]$

$$\frac{1}{2}log\left[\frac{1+x}{1-x}\right] = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$$

$f(x) = \frac{1}{2}log\left[\frac{1+x}{1-x}\right]$

(ii) Now, $f\left(\frac{2x}{1+x^2}\right) = \frac{1}{2}log\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right)$

$= \frac{1}{2}log\left(\frac{1+x^2+2x}{1+x^2-2x}\right)$

$= \frac{1}{2}log\left(\frac{1+x}{1-x}\right)^2$

= 2 f(x).

11. Sum the series to infinity $\log_3 e - \log_9 e + \log_{27} e - \log_{81} e + \ldots\ldots\ldots\ldots$

Solution.

$\log_3 e - \log_9 e + \log_{27} e - \log_{81} e + \ldots\ldots\ldots\ldots$

$= \frac{1}{log_e 3} - \frac{1}{log_e 9} + \frac{1}{log_e 27} - \frac{1}{log_e 81} + \cdots$

$= \frac{1}{log_e 3} - \frac{1}{2log_e 3} + \frac{1}{3log_e 3} - \frac{1}{4log_e 3} + \cdots$

$= \frac{1}{log_e 3}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$

$= \frac{log_e 2}{log_e 3} = log_e 2 \times log_3 e = log_3 2.$

12. Show that $(1 + x)^{1 + x} = 1 + x + x^2 + \frac{1}{2}x^3$ neglecting and higher powers of x. Also find an approximate value of $(1.01)^{1.01}$.

Solution.

$(1 + x)^{1 + x} = e^{log\,(1+x)^{1+x}}$

$= e^{(1+x)log\,(1+x)}$

$\approx e^{(1+x)\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3\right)}$

$\approx e^{x+\frac{1}{2}x^2-\frac{1}{6}x^3}$

$\approx 1 + \left(x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right) + \frac{1}{2!}\left(x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right)^2 + \frac{1}{3!}\left(x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right)^3$

$\approx 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{2!}(x^2 + x^3) + \frac{1}{3!}x^3$

$\approx 1 + x + x^2 + \frac{1}{2}x^3$

Put x = .01 in the result.

$(1.01)^{1.01} = 1 + .01 + .0001 + \frac{1}{2}(.000001) = 1.0101005.$

13. Prove $S = \frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \cdots = 2 - log2.$

Solution.

Here $T_n = \frac{1}{n(2n+1)}$

$T_n = \frac{A}{n} + \frac{B}{2n+1}$

We can find A = 1; B = - 2

Therefore $T_n = \frac{1}{n} - \frac{2}{2n+1}$ ....(1)

Putting n = 1, 2, 3, ... in (1) we get

$T_1 = \frac{1}{1} - \frac{2}{3}$

$T_2 = \frac{1}{2} - \frac{2}{5}$

$T_3 = \frac{1}{3} - \frac{2}{7}$

... ...

Therefore $S = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$

$= 1 - \left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$

= 1 – [log 2 – 1]

= 2 – log 2.

14. Prove $S = \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \cdots$ = log 4 – 1

Solution.

$T_n = (-1)^{n-1}\left[\frac{1}{n(n+1)}\right]$

We have $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$T_n = (-1)^{n-1}\left[\frac{1}{n} - \frac{1}{n+1}\right]$ ....(1)

Putting n = 1, 2, 3, ...in (1) we get

$T_1 = \frac{1}{1} - \frac{1}{2}$

$T_2 = -\frac{1}{2} + \frac{1}{3}$

$T_3 = \frac{1}{3} - \frac{1}{4}$

… … … …

… … … …

Therefore S = 1 + 2 $\left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right)$

= 1 + 2 (log 2 – 1)

= log 4 – 1.

15. Prove that log $\left(1 + \frac{1}{n}\right)^n = 1 - \frac{1}{2(n+1)} - \frac{1}{2.3(n+1)^2} - \frac{1}{3.4(n+1)^3} - \cdots$

Solution.

Put $\frac{1}{n+1} = x$

Therefore R.H.S = $1 - \frac{1}{2}x - \frac{1}{2.3}x^2 - \frac{1}{3.4}x^3 - \cdots$

= $1 - \left(1 - \frac{1}{2}\right)x - \left(\frac{1}{2} - \frac{1}{3}\right)x^2 - \left(\frac{1}{3} - \frac{1}{4}\right)x^3 - \cdots$

= $\left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots\right) + \left(1 + \frac{1}{2}x + \frac{1}{3}x^2 + \cdots\right)$

= $-\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right) + \frac{1}{x}\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right)$

= log (1 – x) – $\frac{1}{x}$log(1 – x)

= $\left(1 - \frac{1}{x}\right)$ log (1 – x)

= (1 – n – 1)log$\left(1 - \frac{1}{n+1}\right)$

= – n log$\left(\frac{n}{n+1}\right)$ = log$\left(\frac{n+1}{n}\right)^n$

= L.H.S.

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